

Ninth-Grade Students' Conceptual Understanding of the Number Line

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Abstract

Sixty (35 girls and 25 boys) 9th-grade students' conceptual understanding of the number line was qualitatively assessed through verbal explanations and visual representations. The assessment included an open-ended question focused on students' number line descriptions and the explanations coalesced around six features: sequential ordering (i.e., numbers are sequentially represented), positivity-negativity of numbers (i.e., the number line contains positive and negative numbers), non-centrality (i.e., zero does not have to be in the center), infinity, increment flexibility (i.e., number line increments can vary), and continuity (i.e., numbers can be placed anywhere between minus infinity and plus infinity without breaks). The students' explanations show that these six features emerge in five successive stages in the conceptual understanding of the number line. These stages are (1) no knowledge, (2) sequential ordering and positivity-negativity, (3) infinity and non-centrality, (4) incremental flexibility, and (5) continuity. The last two stages were not found in most descriptions. The results suggest that students' understanding of the number line is incomplete and may be overestimated by commonly used quantitative assessments of number line knowledge.

Keywords

number line, mathematics, numbers, infinity, increments, continuity

Highlights

- The qualitative study examined students' conceptual understanding of the number line.
- Six features emerged in students' number line descriptions.
- These six features manifest in five successive stages in the conceptual understanding of the number line.

Knowledge of the number line is a foundational component of students' mathematical development (e.g., Booth & Siegler, 2006; Geary et al., 2008; Gunderson et al., 2012; Schneider et al., 2018), and performance on number line tasks predicts students' concurrent and later mathematics achievement (Gunderson et al., 2012; Hansen et al., 2015; Jordan et al., 2013; Siegler et al., 2011). Students' accuracy at placing whole and rational numbers on a physical line is thought to reflect their understanding of the corresponding magnitudes (Siegler & Braithwaite, 2017). The importance of the latter and the utility of the number line in predicting this knowledge has motivated research into the brain and cognitive systems that support students' ability to represent and understand the number line (Geary et al., 2021; Gersib et al.,

2024; Lunardon et al., 2023; MacKay et al., 2023; Opfer & Siegler, 2007; Rouder & Geary, 2014; Siegler & Booth, 2004; Siegler & Opfer, 2003; Zorzi et al., 2002).

Most previous studies have focused on how students map numerals onto a physical number line and developmental change in this competence (for a review, see Schneider et al., 2017), but, as far as we know, have not explored how students think about and visually represent the number line. The current study filled this knowledge gap through a qualitative assessment of ninth-grade students' verbal and visual descriptions of the number line. These qualitative findings highlight components of students' number line knowledge that are not captured by standard quantitative methods (i.e., numeral-to-line placement accuracy).

The Number Line

The number line can be described as a horizontal straight line containing all real numbers, including whole numbers, rational numbers, and irrational numbers, which are sequentially taught over the school years. The number line serves as an effective tool to teach the number system and associated properties and operations (e.g., Viegut & Matthews, 2023) and evaluate students' comprehension of the quantities represented by numbers (Opfer & Siegler, 2007). It is introduced in early elementary school and continues to be used in various aspects of mathematics (e.g., coordinate plane) through middle and high school (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010).

Theoretical Framework

The context in which the number line is used changes across schooling and potentially (re)shapes students' number line models. No studies have examined how contextual changes are expressed in students' number line conceptualizations. In this study, we address this gap by examining how students describe the number line following Vosniadou's (2021) conceptual framework theory. The latter provides an explanation of how a concept is formed in students' minds and how it changes with the addition of newly acquired knowledge.

We also summarize several theoretical approaches for explaining how students acquire an understanding of numbers and numerical magnitude, given magnitude representation is often assessed using the number line. These theories include the integrated theory of numerical development, the symbol+ hypothesis, and the analog+ hypothesis (Fischer, 2003; Siegler, 2016; Varma et al., 2019).

Concepts Associated With the Number Line

As mentioned above, it is important to examine the various contexts in which students encounter the number line, as these will likely influence their conceptual understanding of the number line. According to Vosniadou and colleagues, students' initial understanding of a concept can change as students are exposed to new ways and contexts in which the concept is relevant (Vosniadou, 2021; e.g., Vosniadou & Brewer, 1992), which likely applies to their understanding of the number line. This is because the number line is used in many different mathematical contexts, and the range of numbers represented on the line expands across schooling.

Students' number line conceptualization begins with their exposure to numbers and magnitude comparison. During the elementary years, students learn numbers have a magnitude that corresponds to a location on the physical number line, beginning with whole and later expanding to rational numbers (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). Gains in students' understanding of how the magnitude of numbers is positioned on the number line, reflecting their general understanding of the numerical magnitude, has been the subject of numerous empirical and theoretical studies by educational and cognitive scientists such as Siegler (2016), Tian and Siegler (2018), and Vamvakoussi and Vosniadou (2004, 2007).

The integrated theory of numerical development is one of the most prominent approaches in this area. The theory focuses on how non-symbolic quantities (e.g., discriminating two from three objects) affect students' understanding of the quantities represented by small number words (e.g., three) and numerals (e.g., vanMarle et al., 2018) and how this early knowledge and schooling contribute to their emerging understanding of the quantities represented by whole numbers and eventually rational numbers (Siegler, 2016). In these studies, the number line is used to quantitatively

assess students' knowledge of numerical magnitude. Although the number line is a measurement tool in this line of research, the studies indicate that knowledge of the number line itself is an important component of mathematical development. This is because students' accuracy in placing numbers on the number line is consistently related to their concurrent and later mathematics achievement (Schneider et al., 2018). In other words, students' understanding of the numerical magnitudes represented by different positions on the number line is an important part of their conceptual understanding of the numerical magnitude, although the process is protracted and difficult for many students (Siegler & Braithwaite, 2017; Vamvakoussi & Vosniadou, 2010).

This developmental process might manifest itself in students' number line conceptualization. Many students may intrinsically understand that the numbers progress sequentially on the number line as they learn whole and rational numbers. Meanwhile, they might notice that the increments might vary depending on the context. For example, it could be 1 or 2 while representing whole numbers between 0 and 100 and $1/5$ or $1/10$ while showing rational numbers between 0 and 1. This process might be through assimilation of the initial emphasis on equal increments in the number line (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010).

In the following years, students learn integers. There are two current views that focus explicitly on the mental representations of integers on the number line (Vest & Alibali, 2021). On the one hand, the symbol+ hypothesis posits that negative numbers are understood using symbols, including positive and negative signs. For instance, -4 is conventionally understood as being less than $+3$ due to this symbol convention. (Fischer, 2003; Varma et al., 2019). On the other hand, the analog+ hypothesis proposes that negative numbers represent a continuum of positive numbers on which zero is in the middle as the symmetry point, and the integration of negative integers occurs by reflecting positive numbers through zero on the number line (Varma et al., 2019). The former hypothesis implies negative numbers are discrete symbols, while the latter suggests negative numbers are values opposite in direction to positive numbers, reflected through zero. Although there were differences among these approaches, both are associated with the positivity and negativity of the numbers, which possibly have some correspondence on students' number line conceptualization.

Meanwhile, experimental tasks that compare positive numbers to negative integers, such as reaction time experiments, have been used to test both approaches (Vest & Alibali, 2021). One such parameter is the *distance effect*, which indicates that a greater distance between two natural numbers results in a shorter reaction time when comparing their magnitudes (e.g., when determining which is larger, 4 vs. 25 is faster than comparing 4 vs. 5). Vest and Alibali (2021) observed an *inverse distance effect* (e.g., comparing 4 vs. -5 is faster than comparing 4 vs. -25) when participants compared negative integers to positive integers, supporting the analog+ hypothesis. Although these results provide valuable insights into students' understanding of integers, they may not fully capture the nuances of this understanding (Vest & Alibali, 2021). Further, the representations of integers on the number line in school textbooks are mostly in line with the analog+ hypothesis. Zero is located at the center of the number line and is introduced as a starting point. That might affect students' number line conceptualization. A mature understanding of the numerical magnitude of integers and their correspondence on the number line might involve representations of zero and negative numbers, an area that warrants further exploration.

After learning integers, students transition to more abstract numbers, which, according to Vosniadou's (2021) model, should result in qualitative changes in students' understanding of the number line, although this has not been thoroughly assessed. In current studies, students' acquisition of more abstract number knowledge is explained through referential models, specifying that students use concrete number knowledge as a referent in placing more abstract numbers. For instance, perfect squares in the form of radical expressions (e.g., $\sqrt{9}$) are used as a reference point to place irrational numbers (e.g., $\sqrt{10}$) on the number line (Patel & Varma, 2018). However, studies of students' placement of more abstract numbers on the number line are limited. For instance, students' representations of irrational numbers (e.g., π) on their number line drawings have not been systematically explored. Their manifestation might also be associated with flexible increment choices on the number line.

Further, students are exposed to the number line in other mathematics topics. In the elementary years, they use the number line to illustrate patterns and perform arithmetic operations (e.g., Gonsalves & Krawec, 2014; Herbst, 1997; Saxe et al., 2013; Sutherland et al., 2024). As they progress through middle and high school, they learn more advanced topics, such as algebra, in which they use the number line to describe a particular range of numbers as a solution to problems (e.g., Dickinson & Eade, 2004; McCabe et al., 2010). For instance, they use the intervals in relation to more advanced

concepts, such as inequalities (e.g., given $|x-2|>2$, what are the intervals of x ?), where the intervals or line segments are not necessarily partitioned using equal increments. As a result, they might have a more flexible understanding of the number line by working with the segments of the number line (e.g., intervals). Indeed, this is an important aspect of students' number and number line development but has not been explored qualitatively.

Infinity

Although infinity is not a specific topic in the mathematics curriculum, it is an important aspect of the number line. Students' initial understanding of infinity typically emerges in children around the age of 6-8 years, focusing on natural numbers, and expands to rational numbers around the fifth or sixth grade (Singer & Voica, 2008). As students develop their number knowledge and represent the magnitudes on the number line, they are exposed to the fact that the numbers represented by the number line are boundless for both positive and negative values; that is, the number line can be extended indefinitely, as indicated by arrows. The infinity concept associated with the number line also exists in advanced mathematics (Kleiner, 2001; McDonald & Brown, 2008). For instance, solving inequalities and constructing number lines with intervals taught in eighth grade [e.g., given $x < 5$, $x \in (-\infty, 5)$, and the solution set is shown as a solid bold line starting from an unfilled circle on five and extending towards infinity] might be associated with a solid understanding of infinity. Despite this crucial link between infinity and the number line, it is surprising that many primary, middle, and high school mathematics curricula do not address this relationship explicitly (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). It points to further research on how students see the concept of infinity manifesting itself in the number line conceptualization.

Current Study

The primary goal of the current study was to identify the concepts that emerged in an open-ended task that elicited students' verbal and visual descriptions of the number line. As noted, this approach is valuable because it may provide insights into students' comprehension of numerical magnitude and conceptualization of the number line that is not evident using traditional experimental tasks. We focused on students who had just completed ninth grade because they should have learned all core topics associated with number line models by that point. That is, they had recently encountered various number sets, such as integers and irrational numbers, and had been exposed to mathematics problems that required number line use (e.g., algebraic inequalities) (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). Additionally, it was also right after a transition from middle school to high school where they practiced these skills.

Method

Participants

This study recruited a convenience sample of 60 ninth-grade students (35 girls and 25 boys) in Turkey. We contacted high school principals and mathematics teachers, posting on social media and advertising on an experiment website. We kept participation open to all high school students. As a result, students from 40 different high schools participated. All students were native speakers and were between 14 and 17 years old ($M = 15.16$, $SD = 0.54$, range 14.17 to 16.58 yrs.). The demographic information was gathered through online parental surveys. We recorded the education level of the parents to ensure there was no bias due to differences in parental education.

Parents' education levels were diverse: 22% ($n = 13$) of fathers finished primary school, followed by 8% ($n = 5$), 23% ($n = 14$), and 37% ($n = 22$) of fathers completing middle school, high school, and university, respectively. The remaining 10% ($n = 6$) of fathers completed at least one graduate degree. Twenty-five percent of mothers ($n = 15$) completed primary school, followed by 12% ($n = 7$), 33% ($n = 20$), and 23% ($n = 14$), completing middle school, high school, and university, respectively. The remaining 7% ($n = 4$) of mothers had at least one graduate degree. This study has ethical approval from the local institution (IRB# E-84391427-050.01.04-18335/ 2021-31). We conducted the study through an

online meeting platform due to COVID-19 restrictions on in-person assessments, with the consent of legal guardians for online meeting recordings.

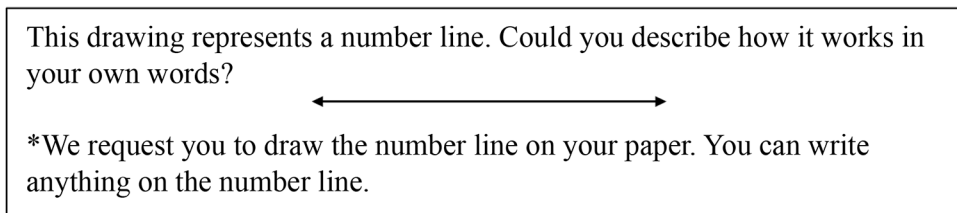
Also, through online surveys, the students' scores on the High School Entrance Exam (LGS, Turkish acronym) were collected to describe the academic characteristics of the sample. This is because students' academic ability could influence their explanation of the number line, and thus, sample characteristics have implications for the generalizability of the results. The sample was of relatively high ability based on their performance on LGS administered to over 1.4 million students in 2020. The exam covers various subjects, including mathematics, science, Turkish, history (revolution history and Kemalism), religion (religious culture and moral knowledge), and foreign language (for all test-takers, $M = 286.35$ out of 500, $SD = 69.22$). Relative to the overall sample of test takers, the mean LGS score for our sample was substantively above average ($M = 384.60$, $SD = 81.51$, $d = 1.30$). In addition, students were specifically asked for the number of correct answers in the mathematics section of the LGS exam. The mathematics questions covered algebra, data statistics, and radical numbers. The participants' average correct responses (for the current sample: $M = 10.35$, $SD = 5.52$) were slightly higher than the average of all students who took the exam ($d = 0.20$). Detailed information about the exam can be found online¹.

Measure

We used a number line question (see Figure 1), which included both written and verbal responses from students. The question asked students to describe in their own words how a number line works and then draw a number line on a blank paper to illustrate their explanation. Additional prompts were included to elicit elaboration on the placement of 0 on the number line and students' understanding of infinity. If the students stated the zero was in the middle of the number line or placed it in the middle, we asked if it had to be in that position. The other prompt focused on students' understanding of infinity on the number line (below). If students did not mention infinity, we prompted what the arrows of the number line represented, possibly revealing knowledge of infinity. The list of questions is in Appendix. The students' responses were recorded through video/audio recordings for analysis. Six students were not prompted on the location of 0, although they reported zero, and one student neither provided infinity as an answer nor got a prompt due to experimental error.

Figure 1

Number Line Question



Note. *The interviewer stated the request verbally.

Procedure

Two researchers conducted individual interviews with students through an online meeting platform due to COVID-19 restrictions; a previous study with ninth graders indicated little difference in mathematical task performance across online and in-person formats (e.g., Geary et al., 2023). One researcher acted as the interviewer, while the other provided technical support. The interviews consisted of open-ended questions and follow-up questions. Although we did not put any time constraints on the students, the average time to complete the interview was four minutes. Prior to the

1) https://osf.io/phfdx/?view_only=824a57c41ca54769a3455c2fcc6ba771

interviews, the researchers provided a detailed explanation of the students' rights, clarified the task requirements, and addressed any general questions the students had.

The interview questions focused on the students' understanding of the number line, including visually represented number lines and related terminology. The researchers encouraged students to elaborate on their answers, particularly if the explanation was unclear (e.g., "You said ... can you elaborate on that?" or "Could you talk a little bit more about ...?"). Students used blank answer sheets to record their responses. At the end of each session, students took photos of their answer sheets. Then, they submitted the photos electronically to the researchers before the Zoom session ended. They also showed their work on the screen, which enabled researchers to ensure that the photo matched the students' actual work. The interview was recorded for later analysis.

Data Analytic Strategies

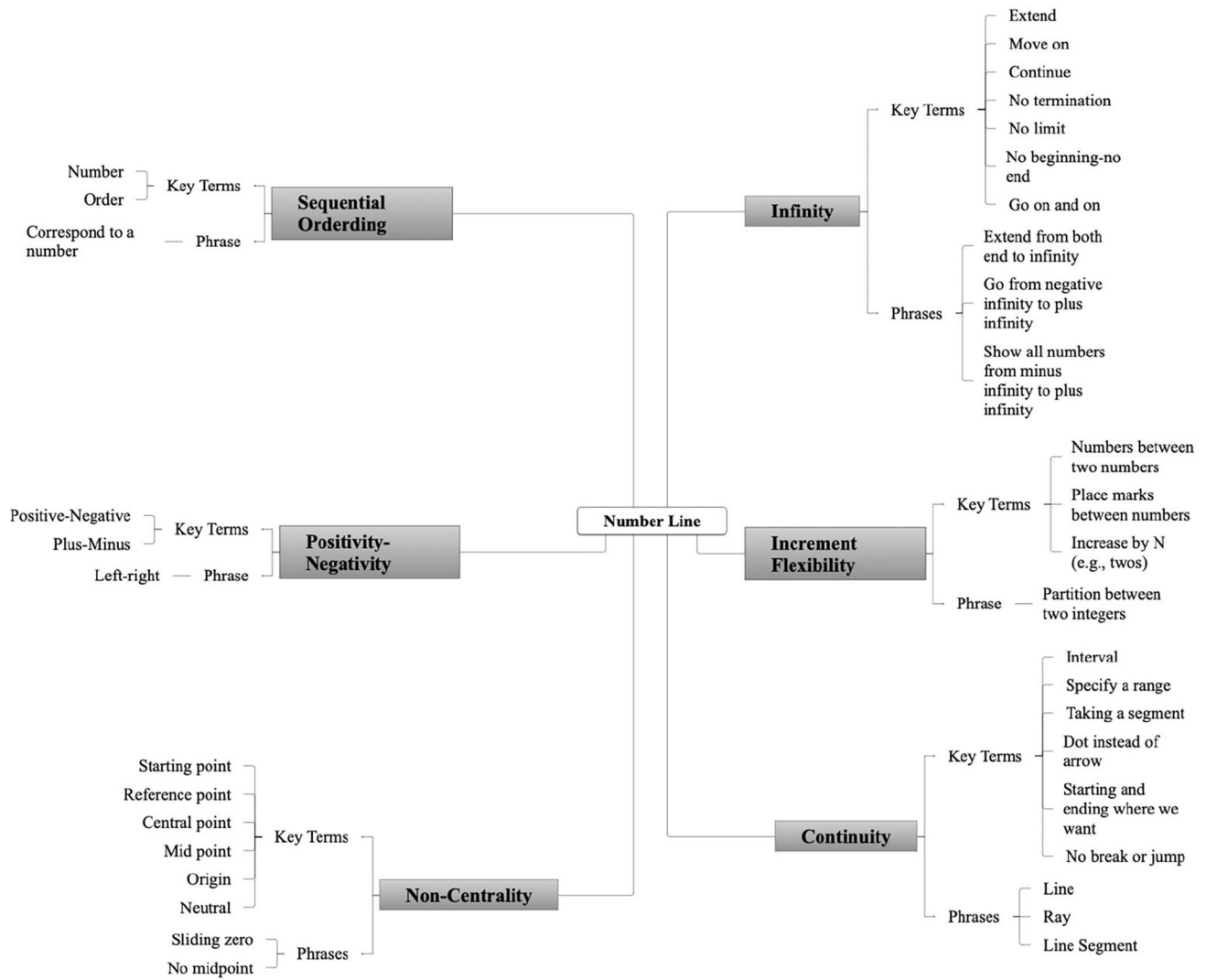
This study employed a qualitative research design, following Kim et al.'s procedures (Kim, Benner, Ongbongan, et al., 2008; Kim, Benner, Takushi, et al., 2008). Transcriptions of interviews were prepared by a trained research assistant and then reviewed and edited by two trained researchers. Later, one of the authors, also a graduate student, listened to and approved the final version of the transcripts. Three graduate students (also authors) analyzed the transcribed data using MAXQDA 2022 (VERBI Software, 2021). This software can be used to mark critical information in data with various codes and organize codes under common themes linked to existing theories, and it enables the development of new theories.

There are various methods to analyze qualitative data (see Boyatzis, 1998). The current study employed Braun and Clarke's (2006) six-phase approach to thematic analysis. This method generated a detailed, reflective, and organized database through the following steps: (1) becoming familiar with the data, (2) producing preliminary low-level codes, (3) exploring themes, (4) checking themes, (5) describing and naming themes, and (6) generating the report. Three researchers reviewed all transcripts and collaborated to identify and name the core themes, with the results outlined by one researcher based on the coding system generated (below) using MAXQDA 2022 (VERBI Software, 2021). Overall, six categories of students' conceptual understanding of the number line emerged from the interviews: *sequential ordering*, *positivity-negativity*, *non-centrality*, *infinity*, *increment flexibility*, and *continuity*. Figure 2 summarizes the key terms and phrases for each category, and detailed explanations are presented in the Results. The coders also noted the number types (e.g., integers, irrational numbers) used in the number line explanations.

The concept of reliability in qualitative studies is different from quantitative studies and is more difficult to define (Golafshani, 2003; Leung, 2015). In qualitative studies, reliability is viewed through meticulousness and quality that can be achieved using various steps (Golafshani, 2003; Johnson et al., 2020). First, we used software (MAXQDA 2022) to label the concepts more systematically. Three of the authors held meetings at every stage and examined all the documents together. They went over the data at least two times at each stage of the coding. While creating the themes, we stipulated that at least two authors had to agree on the same theme. In other words, at least two of the three authors had to agree on the same idea on all themes during the meeting. After the themes were constructed, they sent them to the other authors, and their feedback was taken to ensure that the concepts were clear and concise. Their feedback was used to further improve the concepts.

Figure 2

Key Terms and Phrases

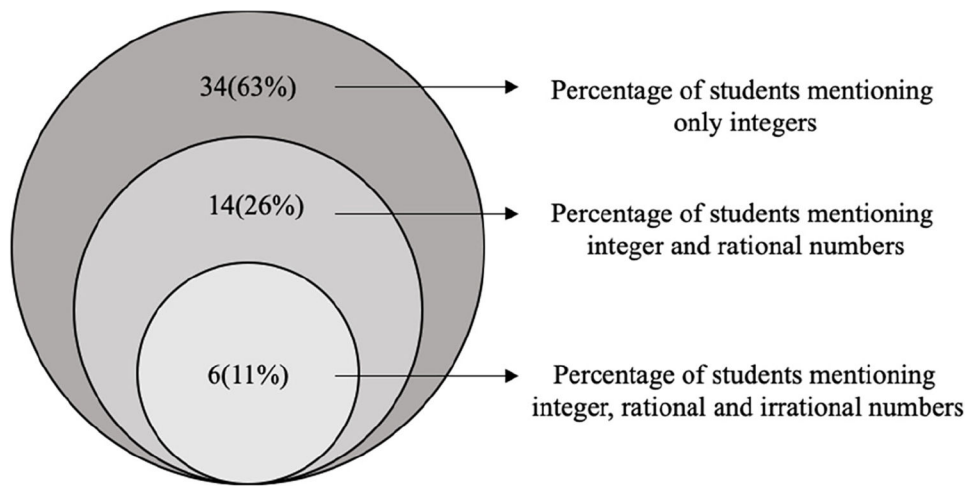


Results

The vast majority ($n = 54, 90\%$) of the students demonstrated proficiency in placing integers on the number line, both positively and negatively. While a minority of students referenced rational ($n = 20, 33\%$) or irrational ($n = 6, 10\%$) numbers, the results revealed a clear hierarchical trend, with those referencing irrational numbers also being adept at rational numbers and integers. Likewise, those who mentioned rational numbers also demonstrated proficiency with integers, as shown in Figure 3.

Figure 3

The Percentage of Number Types Used in Students' Explanations



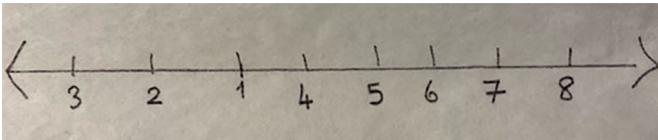
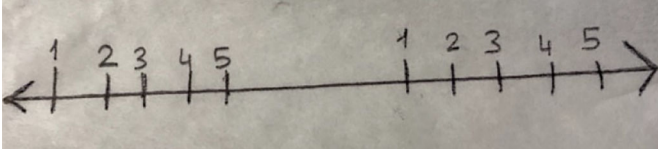
Note. The diagram belongs to the 54 students who included at least integers, including positive and negative numbers, in their number line explanations.

Sequential Ordering

Fifty four of the 60 students referred to the conventional representation of numbers in an ascending or descending sequence on the number line. The six remaining students provided neither verbal nor visual representations of sequential ordering. Table 1 illustrates examples of verbal or visual descriptions provided in incorrect sequences. The pictures were redrawn by researchers due to low resolutions of original student responses.

Table 1

Sequential Ordering Examples

Verbal explanation in English	Original transcript	Note
 <p>S43: "That's all I remember, but we were doing something like this."</p> <p>*I: "I SAW THE RIGHT SIDE REMAINED EMPTY. DO YOU WANT TO ADD SOMETHING?"</p> <p>S43: "Numbers can be written there, too."</p>	<p>Ö43: "Bu kadar hatırlıyorum ben, böyle bir şeyler yapıyorduk ama."</p> <p>*M: "SAĞ TARAFI BOŞ KALMIŞ, EKLEMENK İSTEDİĞİN HERHANGİ BİR ŞEY VAR MI?"</p> <p>Ö43: "Oraya da sayı yazılabilir yine."</p>	<p>The student's whole number placement is not in sequential order.</p>
 <p>*I: "I ALSO SEE SOMETHING ON THE NUMBER LINE, LIKE THERE ARE THE SAME NUMBERS ON THE RIGHT AND LEFT SIDE, AREN'T THERE? DID I SEE WRONG?"</p>	<p>*M: "BİR DE ŞEY GÖRDÜM SAYI DOĞRUSUNUN ÜZERİNDE SAĞ TARAFINDA VE SOL TARAFINDA AYNI RAKAMLAR VAR SANKİ Dİ Mİ? YANLIŞ MI GÖRDÜM?"</p>	<p>None of the dimensions are present. The student's whole number placement is not in sequential order.</p>

Verbal explanation in English	Original transcript	Note
S29: "Right."	Ö29: "Doğru."	
*I: "SO, WHY ARE THE SAME NUMBERS ON THE RIGHT AND LEFT OF THE NUMBER LINE, CAN YOU TELL US A LITTLE MORE ABOUT WHY YOU DRAW LIKE THAT?"	*M: "PEKİ NEDEN SAYI DOĞRUSUNUN SAĞINDA VE SOLUNDA AYNI RAKAMLAR VAR, BİRAZ BAHSEDEBİLİR MİSİN NEDEN ÖYLE ÇİZDİĞİNDEN?"	
S29: "I don't know."	Ö29: "Bilmiyorum."	

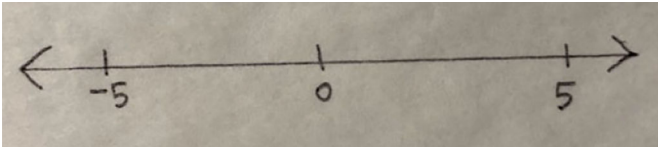
Note. *I/M= the interviewer's talk; S/Ö= the student's responses. Caps indicate the interviewer's talk. The dialogs were shortened to include core ideas (e.g., excluding filler terms, such as uhm).

Positivity-Negativity of Numbers

Fifty-four of the 60 students included both positive and negative numbers when explaining or visually representing the number line. For the visual representations, 25 students (46% of positivity-negativity responders) placed numbers symmetrically; that is, they placed the positive and negative versions of the numbers on the number line (see Table 2 for a related example).

Table 2

Positivity–Negativity Related Example

Verbal explanation in English	Original transcript	Note
 <p>S26: "I put 0 in the middle. Then, I wrote -5 somewhere to the left of the 0. I wrote +5 somewhere on the right side. How is it used? It shows the location of the numbers. That is, it shows that the negative numbers are to the left of the 0, and the positive numbers are to the right of the 0."</p>	<p>Ö26: "Ortaya 0 koydum bir tane. Sonra 0'in sol tarafında bir yere -5 yazdım. Sağ tarafında bir yere de +5 yazdım. Nasıl kullanılıyor? Sayıların yerini gösteriyor. Yani, eksi sayıların 0'in sol tarafında, artı sayıların 0'in sağ tarafında olduğunu gösteriyor."</p>	<p>Negative and positive numbers/sides on the number line. The student used integers. The positive integer has a negative version.</p>

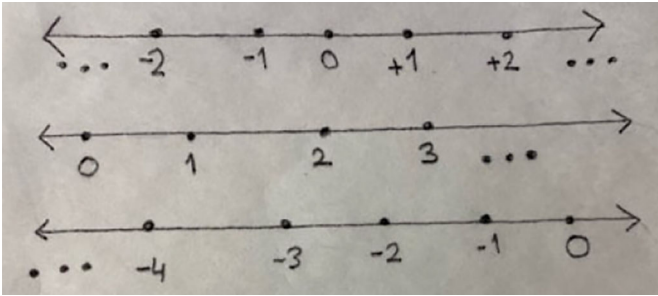
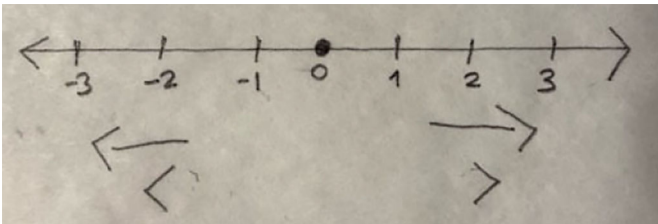
Note. S/Ö = the student's response. The dialogs were shortened to include core ideas (e.g., excluding filler terms, such as uhm).

Non-Centrality

Fifty-one of the 60 students understood that the zero can be placed anywhere on the number line, and 25 of them centered it and placed an equal number of positive and negative numbers to the right and left, respectively. Further, 38 students (63%) stated that the location of zero could change with ($n = 25$) or without ($n = 13$) a prompt (i.e., "Does the location of the zero have to be in the middle?"). Six students did not mention the location of zero, and no prompt was given due to experimental error (see Table 3 for related examples).

Table 3

Non-Centrality Examples

Verbal explanation in English	Original transcript	Note
 <p>S41: "We write 0 in the middle. We express before 0 with minuses, and after 0 with pluses, or we can only move forward from 0 with pluses, from 0 with minuses or so."</p>	<p>Ö41: "Ö: Hani ortasına 0 yazarız. 0'dan öncesini eksilerle, sonrasını artılarla ifade ederiz ya da sadece 0'dan artılarla, 0'dan eksilerle falan da ilerleyebiliriz."</p>	<p>Zero is in the middle, on the right side, and on the left side in three number lines.</p>
 <p>*I: "... DOES 0 ALWAYS HAVE TO BE IN THE MIDDLE OF THE NUMBER LINE?"</p> <p>S44: "Yes."</p>	<p>*M: "0 HER ZAMAN ORTADA OLMAK ZORUNDA MI SAYI DOĞRUSUNDA?"</p> <p>Ö44: "Evet."</p>	<p>Zero is fixed to be in the middle. There is a fixation point to represent zero.</p>

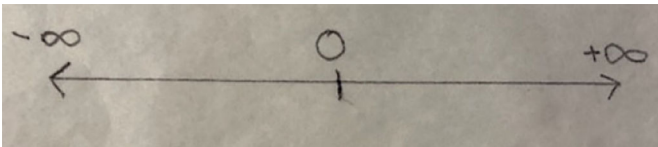
Note. *I/*M= the interviewer's talk; S/Ö = the student's responses. The dialogs were shortened to include core ideas (e.g., excluding filler terms, such as uhm).

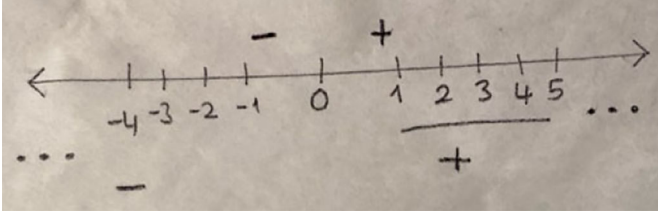
Infinity

The concept of infinity often emerged in 51 of the students' explanations, that is, that the number line could be extended without end. Most students stated this feature of the number line with ($n = 27$) or without ($n = 24$) prompting (i.e., "What do arrows mean?"). Table 4 presents examples of infinity-related explanations. Eleven students had both negative and positive infinity, and 13 of them used symbolic notations, including the infinity symbol near the arrows ($n = 7$; i.e., $\pm\infty$), the interval notation ($n = 3$; i.e., $(-\infty, +\infty)$), and the ellipses ($n = 4$). Moreover, among those placing infinity close to the number line, three placed the symbol on the line as if it were a number.

Table 4

Infinity Examples

Verbal explanation in English	Original transcript	Note
 <p>S11: "It is a line that contains rational and irrational numbers, in other words, all reel numbers, from minus infinity to plus infinity."</p>	<p>Ö11: "Sayı doğrusunun üzerinde rasyonel ve irrasyonel yani tüm reel sayıları bulunduran bir doğrudur, eksi sonsuzdan artı sonsuza."</p>	<p>End-to-end infinity description: from negative infinity to positive infinity. The symbolic notation exists.</p>

Verbal explanation in English	Original transcript	Note
 <p>S20: "There would be a 0 in the middle, so the neutral number, positive numbers on the right side, according to me, and negative numbers on the left side, and left go to infinity unless an interval is specified."</p>	<p>Ö20: "Bunun tam ortasında bir 0 olur, yani nötr sayı bunun bana göre sağ tarafında pozitif sayılar, sol tarafında negatif sayılar olur. Ve sol bir aralık belirtilmediği surece sonsuza kadar gider."</p>	<p>Middle to periphery infinity description: Start from central points and goes to infinity from both ends. Triple dots exist.</p>

Note. S/Ö = the student's responses. The dialogs were shortened to include core ideas (e.g., excluding filler terms, such as uhm).

Furthermore, two distinct descriptions of infinity were observed, one from end-to-end and the other from middle-to-periphery, as depicted in Figure 4. Middle-to-periphery descriptions entail infinity starting from zero and extending in both directions towards the periphery indefinitely (middle-to-periphery, e.g., "It is used to indicate numbers, the midpoint of which is 0, the right side is towards the positive, and the left side is towards the negative, and both sides extend to infinity."). In contrast, the end-to-end descriptions excluded this reference and instead specified numbers from negative/positive infinity to positive/negative infinity (e.g., "The number line exists to represent numbers from negative infinity to positive infinity on a line."). A minority of students ($n = 9$) referred to the number line extending from $\pm \infty$ to $\mp \infty$ (end-to-end), while the majority ($n = 42$) stated that it extends from zero indefinitely in both directions (middle-to-periphery).

Figure 4
Two Types of Infinity Descriptions



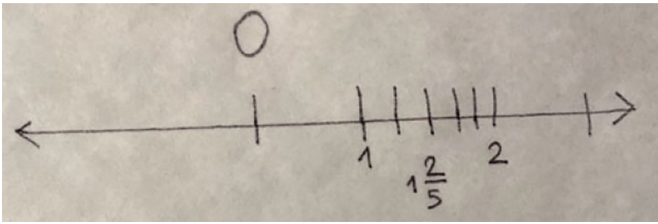
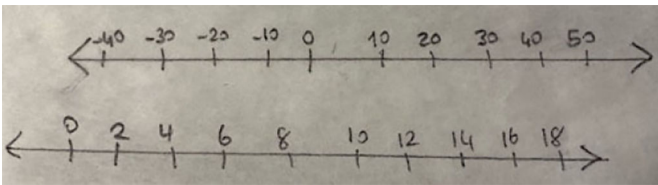
Note. Ellipses represent the direction of movement, with the first line depicting the middle-to-periphery explanation of infinity, where zero is in the middle and the number line extends indefinitely in both directions. The end-to-end description, on the other hand, involves infinity extending from one end to the other; either from negative infinity to positive infinity (represented by a left arrow with ellipsis) or from positive infinity to negative infinity (represented by a right arrow with ellipsis).

Increment Flexibility

Only 20 of the 60 students noted that increments or the divisions on the number line can vary across different portions of the number line or on different number lines. Fourteen of these 20 students visually represented or verbally explained this concept using rational or irrational numbers (e.g., π) placed between whole numbers. The remaining six students parsed the line using only integers ($n = 2$) or demonstrated both approaches ($n = 4$). Table 5 displays two examples, with one having a number line including increments of 1 and 1/5, and the other having two numbers lines with different increments using integers.

Table 5

Increment Flexibility Examples

Verbal explanation in English	Original transcript	Note
 <p>S34: "As far as I remember from primary school, middle school, they used to give a fraction. They asked to find it."</p> <p>*I: "OKAY"</p> <p>S34: "For example, they said one and two fifths."</p>	<p>Ö34: "İlkokuldan, ortaokuldan hatırladığım kadarıyla böyle bir kesirli sayı veriyorlardı. Bunu bulun diyorlardı."</p> <p>*M: "PEKİ"</p> <p>Ö34: "1 tam 2 bölü 5 diyorlardı mesela."</p>	<p>The number line increment varies. Variation is due to rational numbers.</p>
 <p>S19: "If I need a line with increments of 2, I divide it by 2, 2. I divide it by 0, 2, 4. If I need values that are 10s, I divide it by 0, 10, 20, 30, so I don't always need to divide it 1 by 1."</p>	<p>Ö19: "Aralıkları 2 olan bir doğruya ihtiyacım varsa 2, 2 ayırıyorum. 0, 2, 4 diye ayırıyorum. 10 olan değerlere ihtiyacım varsa 0, 10, 20, 30 diye ayırıyorum böyle her zaman 1, 1 diye ayırmaya ihtiyaç duymuyorum."</p>	<p>The number line increment varies. Variation is due to different increments of integers.</p>

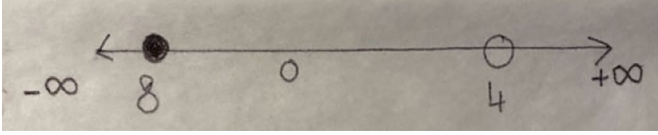
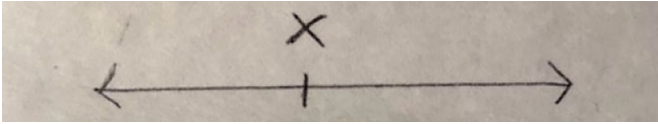
Note. *I/*M= the interviewer's talk; S/Ö= the student's responses. Caps indicate the interviewer's talk. For second example, the numbers on the first number line: -40, -30, -20, -10, 0, 10, 20, 30, 40 and the numbers on the second number line: 0, 2, 4, 6, 8, 10, 12, 14, 16, 18. The dialogs were shortened to include core ideas (e.g., excluding filler terms, such as uhm).

Continuity

Only 15 of 60 students showed an awareness that intervals of numbers can be placed anywhere between minus infinity and plus infinity without abrupt breaks or jumps (see Table 6 for examples). Six of these 15 (40%) students indicated some understanding of continuity (as opposed to 7% of students who did not mention continuity) and used end-to-end descriptions. In other words, end-to-end descriptions were more common among students who used continuity than those who did not.

Table 6

Continuity Examples

Verbal explanation in English	Original transcript	Note
 <p>S31: “We use this to show certain intervals of numbers. For example, the result of an equation is between two specific numbers, or one is certain, and the other goes on forever; we show them on the number line.”</p>	<p>Ö31: “Bunu sayıların belli aralıklarını göstermek için kullanıyoruz. Mesela bir denklemin sonucu belirli iki sayı arasında ya da biri belli diğeri sonsuza kadar uzanıyor; onları sayı doğrusunda gösteriyoruz.”</p>	The student provided a line segment.
 <p>S9: “...I put one x point. Any point on the number line. I don't know the value of this point, but I know that it is on the number line, this is good information. Because that way, I can find the range of this number. Because in this range from negative infinity to positive infinity, that is, x is the element of negative infinite and positive infinite range (student wrote the answer, $x \in (-\infty, +\infty)$).”</p>	<p>Ö9: “Bir tane x noktası koyarım. Sayı doğrusunun üzerinde herhangi bir nokta. Bilmiyorum bu noktanın değerini, ama şunu biliyorum ki sayı doğrusunun üzerinde, bu güzel bir bilgi. Çünkü böylelikle bu bu sayının aralığını bulabilirim. Çünkü bu aralıkta $-\infty$ sonsuzdan $+\infty$ sonsuza yani x elemandır $-\infty$ sonsuz $+\infty$ sonsuz aralığı diyebilirim (öğrenci cevabı yazıyor, $x \in (-\infty, +\infty)$).”</p>	The student used interval representation. The interval is between negative infinity and positive infinity.

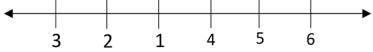

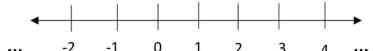


Note. S/Ö = the student's responses. The dialogs were shortened to include core ideas (e.g., excluding filler terms, such as uhm).

Grouping Number Line Responses

We categorized the pattern of students' explanations into five groups using a Guttman (1944) scale to establish a hierarchy. Each concept was coded as included in the student's explanations and were correct or were not included or were incorrect if mentioned. For instance, students in Group A lacked sequential ordering, while those in Group B included it but also did not mention or were incorrect in their explanations of non-centrality or infinity. Students in Group C had accurate infinity and non-centrality, but not incremental flexibility. All students in higher groups accurately mentioned concepts noted in lower groups. Table 7 provides a breakdown of the categories and issues for each group.

Table 7

Descriptions of Concepts and Issues Associated With the Number Line

Group	Concept	Issue	Drawing
A (n = 6)	X	No knowledge No sequential ordering	
B (n = 14)	Positivity-negativity	No non-centrality No infinity	
C (n = 15)	Infinity Non-centrality	No incremental flexibility	
D (n = 10)	Incremental flexibility	No continuity	
E (n = 15)	Continuity	X	

Note. The concept indicates the first time the above-mentioned emerged concept is mentioned. Issue displays the issues observed in the target group’s explanations. Number line examples were grounded on student number line drawings. *n* indicates number of students in each group. The X on the first row indicates that no predetermined concept exists. The X at the bottom row represents that no predetermined issue has been found.

Discussion

The goal of the current study was to analyze ninth graders’ verbal and visual explanations of the number line and to relate these findings to existing models of number development. Six dominant themes emerged from these explanations: *sequential ordering*, *positivity-negativity of numerical magnitudes*, *non-centrality of zero*, *infinity*, *increment flexibility*, and *continuity*. While most students demonstrated an understanding of the sequential placement of positive and negative numbers on a number line, the notion of infinity, and the non-fixed location of zero, fewer students were able to flexibly locate numbers with different increments, parse sections of the line, or describe line segments or intervals as a continuum of numbers. We discuss these results with respect to the conceptual framework theory and current models of number and magnitude development associated with the number line conceptualization, including the integrated theory of numerical development, the symbol+ hypothesis, and the analog+ hypothesis (Fischer, 2003; Siegler, 2016; Varma et al., 2019).

Concepts Associated With the Number Line

The findings suggest a hierarchy among the concepts associated with the number line and are in line with the constructive approach of the conceptual framework theory (Vosniadou, 2021). After students form the number line concept based on whole numbers, they seem to gradually (based on our hierarchy in Table 7) assimilate more complex concepts into their understanding of the line. We will present these concepts in accordance with this hierarchy and group them at the end of this section.

Sequential ordering was the most common feature emerging in ninth graders’ descriptions of the number line. Most students, but surprisingly not all of them, understood that numbers must be situated sequentially on the number line. *Positivity-negativity of numbers* was the next common feature to emerge from the number line explanations, indicating that most of them have at least a basic understanding that the number line contains negative and positive values. Whether it was conveyed verbally or visually, most students referred to integers (including whole numbers, zero, and negative integers), but only a few of them used rational numbers. Further, there was a hierarchical relationship between the discussion of integers and rational numbers, such that all students who used rational numbers always used integers but not the reverse. Even so, most students who described or visually represented rational numbers focused on positive

values, that is, very few of them mentioned negative rational numbers (15% of rational number users and 5% of all students).

Siegler's (2016) integrated number theory focuses on understanding the magnitudes represented by number words and numerals. The model and associated studies show that the learning of the magnitudes represented by whole numbers precedes the learning of the magnitudes represented by rational numbers, but both are eventually integrated into a mental number line (Siegler et al., 2011). Our results suggest that even after years of exposure to rational numbers, integers remain a much more salient feature of students' mental number lines. In other words, the unprompted descriptions of most ninth graders suggest that they do not automatically conceptualize the number line as containing rational numbers; their descriptions and visual depictions remain at a level expected of elementary school students.

This does not mean that they do not understand how positive rational numbers are situated on the number line, as experimental studies indicate that they do understand this (Siegler et al., 2011), but rather integers are much more accessible than rational numbers when students generate their own representation of the number line. The experimental procedures that explicitly ask students to situate rational numbers on the line could overestimate the depth of their understanding of the relationship between rational numbers and the number line. At the very least, our results suggest that many high school students' self-generated representations of the number line remain rudimentary. Further insights were provided by students' comments about irrational numbers. Only a few of them verbally mentioned ($n = 6$) or placed them on the number line ($n = 2$). In line with previous studies, they used integers to situate irrational numbers, emphasizing the use of more concrete numbers to place more abstract ones (Patel & Varma, 2018). This is a reasonable approach but was uncommon.

Furthermore, many students (with or without prompts) stated that the location of zero could vary across contexts, confirming they understood that zero did not need to be centered on the line, referring to the *non-centrality of zero*. More than half of the high school students (63%) seemed to understand and use both the idea of symmetry and the malleability of the location of zero. At the same time, however, a considerable number of student descriptions and illustrations suggested that many believed that zero was a fixed point; that is, it always needed to be represented centrally, even when working with a segment of the line that did not include zero.

When describing or illustrating positive and negative numbers, most students situated zero as a reference point. Many of them stressed the symmetry of the number line (i.e., each positive integer has symmetry with a negative integer, and zero is the center), either verbally stating that every positive number has a negative counterpart or demonstrating the concept in their drawings. Even though most of these students know that zero does not have to be represented at the center of the number line (*non-centrality of zero*), they preferred to use it as a midpoint. These results align with the analog+ hypothesis, proposing that negative integers are placed according to the symmetry concept with zero positioned as the mirror point on the number line and negative integers as the reflections of the representation of positive integers (Varma & Schwartz, 2011; Vest & Alibali, 2021). In other words, as a mathematical entity, zero functions as a reflection point at the center of the number line to determine the placement of the negatives as a reflection of the location of the positives.

The results are also relevant to the prior discussion on the salience of integers over rational numbers; that is, even with an understanding of the non-centrality of zero, nearly all students expressed that using integers. The use of directional explanations of integers could also indicate that students are accessing different features of the number line, in line with Vest and Alibali (2021). In this case, even if students have a good conceptual understanding of the magnitudes represented by integers and rational numbers (Siegler, 2016), their depiction of negative integers could just represent an understanding that negative values are to the left of zero and mirror positive integers without really accessing the magnitude-related implications. In other words, the reflection of whole numbers at zero to create a representation of negative numbers, although accurate, makes it easy to depict them on the number line. However, this in and of itself is not necessarily a strong indication that students conceptually understand negative values.

The concept of infinity also emerged in many students' number line explanations. Students' knowledge about the meaning of the arrows, largely unprompted, on the number line was apparent in their descriptions. However, consistent with Singer and Voica's (2008) findings, considerable variation was observed in the complexity of these descriptions. Many students ($n = 31$) defined infinity only in the context of a subset of numbers, typically integers. In contrast, a few students ($n = 20$) described infinity in terms of a wide range of number sets, including integers and rational and/or

irrational numbers. Further, the use of negative and positive infinity symbols was very rare, indicating that the symbol usage to describe the infinity on the number line did not emerge in the verbal and written descriptions of this age group.

The concept of *increment flexibility* was also observed in some students' number line explanations, that is, the understanding that any segment of the number line can have different increments (e.g., increments of 0.1 and 2: "...it can be used to show an interval between two numbers. For example, between 45 and 46, 45.1 and 45.2. It can be used for a pattern. For example, 1,3,5,7,9,11,13..."). A few students ($n = 2$) drew number lines with different increments, including only integers (e.g., increments by twos and tens), which might be related to the patterning activities or arithmetic operations that students perform in earlier years of schooling (e.g., Gonsalves & Krawec, 2014; Herbst, 1997; Saxe et al., 2013; Sutherland et al., 2024). Still, 14 students placed and/or verbally stated that rational or irrational numbers can be situated between two integers. They stated that the increments could be changed by further dividing the integers, indicating the link between increment flexibility and density-based numbers (i.e., numbers that include infinitely many numbers in between) including rational and irrational numbers (Vamvakoussi & Vosniadou, 2012). Four students did both. In all, only six (10%) and twenty (33%) students used integers and rational numbers, respectively, to describe increment flexibility, suggesting that the remaining students may not fully understand this concept.

Most of these descriptions focused on rational number explanations (some also included irrational numbers), suggesting that incremental flexibility is dependent on a strong understanding of the magnitudes represented by rational numbers, which seems to be related to the last stage of the Integrated Theory of Numerical Development: the expansion of the whole number knowledge to the understanding of rational numbers (Siegler, 2016). The final stage may be multi-faceted. In the beginning, students might be familiar with the magnitudes represented by rational numbers (e.g., $1/2$), whereas later, they might situate them on their mental number line (Geary et al., 2021). Rare use of rational numbers in the context of increment flexibility suggests that many students have had limited integration of rational number knowledge with the number line, even though they have been exposed to rational numbers and increment flexibility since elementary school in various topics (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). These findings are consistent with Vamvakoussi and Vosniadou's (2007) conclusion that acquiring the rational number concept is a slow and continuous process.

Lastly, some students focused on *continuity*, indicating that any segments on the number line could be used to represent a specific number range continuously. This is like increment flexibility, such that the segments being taken could be different (e.g., a line segment between 5 and 10 would be different from a line segment between 2 and 12); however, this differs from increment flexibility because continuity requires an additional understanding of placement of numbers continuously in between the endpoints of the increments. In other words, demonstrating an understanding of line segments is not the same as understanding that, for instance, there are infinite numbers between two consecutive whole numbers (Vamvakoussi & Vosniadou, 2012). Instead, it requires students to understand that any segment could be taken from the number line for a purpose, and numbers are lined up without any breaks.

Continuity is an essential step, as more advanced topics, such as inequalities and piecewise functions, include the notion of continuity. The following problem can be considered as a demonstration: $1 < x \leq 4$. The solution consists of a number line with all numbers in the interval between 1 and 4 (including 4). Such a representation on the number line might be associated with continuity because the solution requires the segmentation of the number line. Additionally, students need to be aware that these segments include numbers continuously.

Returning to the descriptions of infinity, end-to-end description explanations were more common among students who mentioned continuity: 40% of students who stated continuity as opposed to 7% who did not. One potential explanation could be differences in students' perceptions of the number line. Suppose one imagines a rope with rings on it. The former explanation (involving zero as a midpoint referent) requires only two rings hanging in the middle to be sent in opposite directions towards infinity (for visual demonstration, see Figure 4). On the other hand, the latter explanation requires only one ring to slide from one end (e.g., negative infinity) to the other (e.g., positive infinity). The first representation has a break (zero) in the middle, so it might be more difficult for students who have the first number line representation in mind to comprehend continuous relations. Then, we would propose that the end-to-end descriptions could be more compatible with continuity and thus associated with more complicated topics.

The notion of continuity in functions can be considered as an example. It could be stated that "a function is continuous on the interval $[a, b]$ if it is continuous at each point within this interval" (Dawkins, 2018). The zero break

in the middle-to-periphery infinity explanations may impede the idea of continuity in functions. Future research should focus on this issue.

Grouping the Number Line Explanations

As mentioned earlier, the hierarchy appears to exist in concepts associated with the number line that emerged from the students' descriptions. In line with the conceptual framework theory (Vosniadou, 2021), number line concept formation may not be a static process but rather a dynamic one that changes over time with the integration of new mathematics knowledge. As students expand their knowledge and use the number line in novel mathematical contexts, they integrate new number line concepts with existing ones. Considering this, students first understand the sequential ordering and positivity-negativity on the number line (Groups A and B). Subsequently, an understanding of the non-fixed location of zero and infinity associated with the number line emerges (Group C). Students who achieve this level of conceptual understanding appear to be prepared to understand that the number line includes numbers with different increments; that is, for instance, the distance between 0 and 1 can be divided into five pieces with 0.2 increments, whereas the portion between 1 and 2 can remain without partitioning (Group D). That seems to be foundational to the understanding of continuity (Group E). The key finding here is that all students classified at one level showed indications of understanding the core concepts at lower levels.

Educational Implications

There are several educational implications, and before mentioning them, we should note that the number line is one of the tools in mathematics teaching. Some mathematics topics might be easier to understand without using the number line, but even so the use of different representations, including the number line, might enrich mathematics teaching. The results of the study suggest that teachers should revisit the concept of the number line with their students, specifically by explicitly discussing its various features and uses of it. Early elementary education places emphasis on equal increments between whole numbers, particularly when teaching arithmetic operations on the number line (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). However, this can lead to the misconception that the distance between increments is always fixed. This misconception is like other mathematical misconceptions that can develop based on common instructional practices and ways in which mathematics problems are presented in textbooks (Kajander & Lovric, 2009; McNeil et al., 2017; Nathan et al., 2002). Therefore, teachers should refrain from emphasizing fixed-increments of the number line and address a potential misconception by highlighting non-fixed increments while teaching rational numbers, using earlier fixed-increment number lines as a transfer point.

Furthermore, a significant number of students believed that the location of zero on the number line was fixed. This misunderstanding is also likely due to early instruction and textbook presentations (e.g., Saxe et al., 2013; Özdemir, 2022) that center zero on the number line. Also, the number line drawings in books are only on the horizontal line, and they are often prototypes in the curriculum and teacher drawings like that. However, there are other number lines, such as curvy, vertical, or tilted (e.g., rotated 45°). These different representations can be shown to students so that they realize that the number line is just a prototype. That might help them use the number line more flexibly.

Still, this misconception can cause problems when teaching algebraic inequalities, where the location of zero needs to be adjusted. To address this issue, teachers can provide explicit instruction on the location of zero by, for instance, presenting three different number line representations for different algebraic inequalities (e.g., $-5 < x < 5$, $x < -5$, and $-1 < x < 10$). The solution to the first question places zero in the center of the number line, while the solution to the second question omits zero altogether, and the third question has a representation where zero is not centrally located. By showcasing these distinct number line representations, teachers may be able to dispel any misconceptions of a fixed location of zero and ensure their students understand the dynamic and shifting nature of the zero point in mathematical contexts.

Finally, the study found that more than half of the students did not mention continuity on the number line, suggesting that many students have not yet grasped how continuous relations can be represented on the line. Hence, teachers need to emphasize the conceptual change from discrete to continuous when introducing advanced mathematics

knowledge to their students (Vamvakoussi & Vosniadou, 2012). By doing so, they can help students understand how continuous relations are represented on the number line and avoid any misconceptions that may arise.

Limitations

While this study provided valuable insights into high school students' conceptual understanding of the number line, it is important to acknowledge some limitations of the procedure used. The assessment consisted of only one item, which can lead to poor test reliability and less validity in interpreting the results. For instance, some students' understanding of rational numbers might have been underestimated since their descriptions and visual representations were mainly focused on integers. However, experimental studies have shown that most students of this age have some understanding of the magnitudes represented by positive rational numbers (Geary et al., 2021; Siegler et al., 2011). This might be related to the disadvantages of prototype teaching without allowing students self-construction. The mathematics skills or knowledge gained through prototyping (e.g., Simon & Cox, 2019), in this case, the use of the number line as a prototype to teach numbers, might not be generalized across different situations. For example, students might place a rational number on the number line but struggle to construct a number line with rational numbers on it. Additionally, the sample used in this study was well above average in academic achievement and slightly above average in math performance, which may overestimate the typical ninth grade students' understanding of the number line.

Further, students might reproduce or summarize what they learned from math textbooks rather than demonstrate their own self-generated conceptual understanding in their responses. The latter, as in this study, suggests that many students appear to have limited knowledge of how the number line can be used despite exposure to the number line in many mathematical contexts. Also, a standardized protocol was followed for all students, but there was an experimental error while asking the prompting questions for a few of them. As a result, six students did not mention where the zero should be, and they were not prompted due to the error. This could potentially affect the accuracy of the results related to zero.

In addition, due to COVID-19, we collected the data online via Zoom. Therefore, students' answers may differ from those in a face-to-face interview. However, we took several precautions while conducting the interviews to ensure the research was reliable. First, we recorded all the Zoom interviews. Further, we asked the students to hold their written answers on the screen and got students' written work before we talked about their answers. Also, we asked students to send photos of their answers electronically before finishing the interview. In addition, if students' responses were unclear, we asked the students to elaborate on their answers. Lastly, during the interview, there was a second researcher who was always a backup in case of any situation.

Despite these limitations, this study is the first to focus on high school students' self-generated conceptual understanding of the number line, which provides more nuanced insights than experimental procedures alone. The findings have important implications for the assessment of number line conceptualization and for targeted number line interventions aimed at teaching advanced mathematics that require the use of the number line (e.g., Olsen, 1995; Psycharis et al., 2009). As such, this study contributes to a deeper understanding of students' understanding of the number line and can inform more effective mathematics instruction in the classroom.

Conclusion

In conclusion, the study sheds light on the self-generated conceptual understanding of the number line among high school students, indicating that many students have misconceptions regarding the fixed increments and location of zero on the number line. The study suggests that teachers need to revisit the number line and explicitly discuss its different features and uses, emphasizing non-fixed increments while teaching rational numbers and the location of zero when teaching algebraic inequalities. Moreover, the study highlights the importance of emphasizing the conceptual change of the number line from discrete to continuous when introducing advanced mathematics knowledge. While the study has limitations, including a sample of high-achieving students, it provides valuable insights into number line development that can inform the assessment and targeted interventions for improving students' understanding of the number line in mathematics education.

The main goal of the study was to analyze how students conceptualized the number line. The study was exploratory in nature, and it would be useful to investigate how this understanding relates to other mathematical concepts in future research. We need to explore how the number line categories that emerged in this study connect with other mathematical competencies and how we can use this connection to improve learning outcomes.

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Data Availability: Data will be available upon request.

References

- Booth, J. L., & Siegler, R. S. (2006). Developmental and individual differences in pure numerical estimation. *Developmental Psychology*, 42(1), 189–201. <https://doi.org/10.1037/0012-1649.41.6.189>
- Boyatzis, R. E. (1998). *Transforming qualitative information: Thematic analysis and code development*. SAGE.
- Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology*, 3(2), 77–101. <https://doi.org/10.1191/1478088706qp063oa>
- Dawkins, P. (2018, May 29). Section 2–9: Continuity. *Paul's Online Notes*. Retrieved June 8, 2022, from <https://tutorial.math.lamar.edu/classes/calci/continuity.aspx>
- Dickinson, P., & Eade, F. (2004). Using the number line to investigate the solving of linear equations. *For the Learning of Mathematics*, 24(2), 41–47.
- Fischer, M. H. (2003). Cognitive representation of negative numbers. *Psychological Science*, 14(3), 278–282. <https://doi.org/10.1111/1467-9280.03435>
- Geary, D. C., Hoard, M. K., Nugent, L., & Byrd-Craven, J. (2008). Development of number line representations in children with mathematical learning disability. *Developmental Neuropsychology*, 33(3), 277–299. <https://doi.org/10.1080/87565640801982361>
- Geary, D. C., Hoard, M. K., Nugent, L., & Ünal, Z. E. (2023). Sex differences in developmental pathways to mathematical competence. *Journal of Educational Psychology*, 115(2), 212–228. <https://doi.org/10.1037/edu0000763>
- Geary, D. C., Scofield, J. E., Hoard, M. K., & Nugent, L. (2021). Boys' advantage on the fractions number line is mediated by visuospatial attention: Evidence for a parietal-spatial contribution to number line learning. *Developmental Science*, 24(3), Article e13063. <https://doi.org/10.1111/desc.13063>
- Gersib, J. A., Rojo, M., Ghafaghazi, S., Uy, J., & Doabler, C. T. (2024). Measuring success: Integrating number lines into measurement instruction for students with learning disabilities. *Intervention in School and Clinic*, 59(3), 173–182. <https://doi.org/10.1177/10534512231156883>
- Golafshani, N. (2003). Understanding reliability and validity in qualitative research. *Qualitative Report*, 8(4), 597–607.
- Gonsalves, N., & Krawec, J. (2014). Using number lines to solve math word problems: A strategy for students with learning disabilities. *Learning Disabilities Research & Practice*, 29(4), 160–170. <https://doi.org/10.1111/ldrp.12042>
- Gunderson, E. A., Ramirez, G., Beilock, S. L., & Levine, S. C. (2012). The relation between spatial skill and early number knowledge: The role of the linear number line. *Developmental Psychology*, 48(5), 1229–1241. <https://doi.org/10.1037/a0027433>
- Guttman, L. (1944). A basis for scaling qualitative data. *American Sociological Review*, 9(2), 139–150. <https://doi.org/10.2307/2086306>
- Hansen, N., Jordan, N. C., Fernandez, E., Siegler, R. S., Fuchs, L., Gersten, R., & Micklos, D. (2015). General and math-specific predictors of sixth-graders' knowledge of fractions. *Cognitive Development*, 35, 34–49. <https://doi.org/10.1016/j.cogdev.2015.02.001>
- Herbst, P. (1997). The number-line metaphor in the discourse of a textbook series. *For the Learning of Mathematics*, 17(3), 36–45.

- Johnson, J. L., Adkins, D., & Chauvin, S. (2020). A review of the quality indicators of rigor in qualitative research. *American Journal of Pharmaceutical Education*, 84(1), Article 7120. <https://doi.org/10.5688/ajpe7120>
- Jordan, N. C., Hansen, N., Fuchs, L. S., Siegler, R. S., Gersten, R., & Micklos, D. (2013). Developmental predictors of fraction concepts and procedures. *Journal of Experimental Child Psychology*, 116(1), 45–58. <https://doi.org/10.1016/j.jecp.2013.02.001>
- Kajander, A., & Lovric, M. (2009). Mathematics textbooks and their potential role in supporting misconceptions. *International Journal of Mathematical Education in Science and Technology*, 40(2), 173–181. <https://doi.org/10.1080/00207390701691558>
- Kim, S. Y., Benner, A. D., Ongbongan, K., Acob, J., Dinh, K. T., Takushi, R. M. N., & Dennerlein, D. (2008). Children of Filipino immigrants in Hawai'i: Adolescent girls' experiences at home and at school. *Journal of Immigrant & Refugee Studies*, 6(4), 591–598. <https://doi.org/10.1080/15362940802480670>
- Kim, S. Y., Benner, A. D., Takushi, R. M. N., Ongbongan, K., Dennerlein, D., & Spencer, D. K. (2008). "It's like we're just renting over here": The pervasive experiences of discrimination of Filipino immigrant youth gang members in Hawai'i. *AAPI Nexus*, 6(1), 11–30. https://doi.org/10.36650/nexus6.1_11-30_KimEtAl
- Kleiner, I. (2001). History of the infinitely small and the infinitely large in calculus. *Educational Studies in Mathematics*, 48(2-3), 137–174. <https://doi.org/10.1023/A:1016090528065>
- Leung, L. (2015). Validity, reliability, and generalizability in qualitative research. *Journal of Family Medicine and Primary Care*, 4(3), 324–327. <https://doi.org/10.4103/2249-4863.161306>
- Lunardon, M., Lucangeli, D., Zorzi, M., & Sella, F. (2023). Math computerized games in the classroom: A number line training in primary school children. *Progress in Brain Research*, 276, 1–33. <https://doi.org/10.1016/bs.pbr.2022.11.001>
- MacKay, K. J., Germeys, F., Van Dooren, W., Verschaffel, L., & Luwel, K. (2023). EXPRESS: The structure of the notation system in adults' number line estimation: An eye-tracking study. *Quarterly Journal of Experimental Psychology*, 76(3), 538–553. <https://doi.org/10.1177/17470218221094577>
- McCabe, T., Sorto, M. A., & White, A. (2010). Activities for students: Algebra on the number line. *Mathematics Teacher*, 104(5), 379–386. <https://doi.org/10.5951/MT.104.5.0379>
- McDonald, M. A., & Brown, A. (2008). Developing notions of infinity. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics education* (pp. 55–64). Mathematical Association of America.
- McNeil, N. M., Hornburg, C. B., Fuhs, M. W., & O'Rear, C. D. (2017). Understanding children's difficulties with mathematical equivalence. In D. C. Geary, D. B. Berch, R. J. Ochsendorf, & K. M. Koepke (Eds.), *Acquisition of complex arithmetic skills and higher-order mathematics concepts* (pp. 167–195). Academic Press. <https://doi.org/10.1016/B978-0-12-805086-6.00008-4>
- Nathan, M. J., Long, S. D., & Alibali, M. W. (2002). The symbol precedence view of mathematical development: A corpus analysis of the rhetorical structure of textbooks. *Discourse Processes*, 33(1), 1–21. https://doi.org/10.1207/S15326950DP3301_01
- National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common Core State Standards for Mathematics*. https://doi.org/10.1207/S15326950DP3301_01
- Olsen, J. R. (1995). The effect of the use of number lines representations on student understanding of basic function concepts. In D. T. Owens (Ed.), *Proceedings of the annual meeting of the North American chapter of the international group for the psychology of mathematics education* (Vol. 2, pp. 308–312). ERIC Clearinghouse.
- Opfer, J. E., & Siegler, R. S. (2007). Representational change and children's numerical estimation. *Cognitive Psychology*, 55(3), 169–195. <https://doi.org/10.1016/j.cogpsych.2006.09.002>
- Özdemir, E. (2022). An investigation of prospective mathematics teachers' ability to subtract integers with a number line and counters. *Kastamonu Eğitim Dergisi*, 30(1), 205–216. <https://doi.org/10.24106/kefedergi.833879>
- Patel, P., & Varma, S. (2018). How the abstract becomes concrete: Irrational numbers are understood relative to natural numbers and perfect squares. *Cognitive Science*, 42(5), 1642–1676. <https://doi.org/10.1111/cogs.12619>
- Psycharis, G., Latsi, M., & Kynigos, C. (2009). Meanings for fraction as number measure by exploring the number line. *The International Journal for Technology in Mathematics Education*, 16(3), 91–107.
- Rouder, J. N., & Geary, D. C. (2014). Children's cognitive representation of the mathematical number line. *Developmental Science*, 17(4), 525–536. <https://doi.org/10.1111/desc.12166>
- Saxe, G. B., Diakow, R., & Gearhart, M. (2013). Towards curricular coherence in integers and fractions: A study of the efficacy of a lesson sequence that uses the number line as the principal representational context. *ZDM*, 45(3), 343–364. <https://doi.org/10.1007/s11858-012-0466-2>

- Schneider, M., Beeres, K., Coban, L., Merz, S., Susan Schmidt, S., Stricker, J., & De Smedt, B. (2017). Associations of non-symbolic and symbolic numerical magnitude processing with mathematical competence: A meta-analysis. *Developmental Science*, *20*(3), Article e12372. <https://doi.org/10.1111/desc.12372>
- Schneider, M., Merz, S., Stricker, J., De Smedt, B., Torbeyns, J., Verschaffel, L., & Luwel, K. (2018). Associations of number line estimation with mathematical competence: A meta-analysis. *Child Development*, *89*(5), 1467–1484. <https://doi.org/10.1111/cdev.13068>
- Siegler, R. S. (2016). Magnitude knowledge: The common core of numerical development. *Developmental Science*, *19*(3), 341–361. <https://doi.org/10.1111/desc.12395>
- Siegler, R. S., & Booth, J. L. (2004). Development of numerical estimation in young children. *Child Development*, *75*(2), 428–444. <https://doi.org/10.1111/j.1467-8624.2004.00684.x>
- Siegler, R. S., & Braithwaite, D. W. (2017). Numerical development. *Annual Review of Psychology*, *68*(1), 187–213. <https://doi.org/10.1146/annurev-psych-010416-044101>
- Siegler, R. S., & Opfer, J. E. (2003). The development of numerical estimation: Evidence for multiple representations of numerical quantity. *Psychological Science*, *14*(3), 237–243. <https://doi.org/10.1111/1467-9280.02438>
- Siegler, R. S., Thompson, C. A., & Schneider, M. (2011). An integrated theory of whole number and fractions development. *Cognitive Psychology*, *62*(4), 273–296. <https://doi.org/10.1016/j.cogpsych.2011.03.001>
- Simon, L. M., & Cox, D. C. (2019). The role of prototyping in mathematical design thinking. *The Journal of Mathematical Behavior*, *56*, Article 100724. <https://doi.org/10.1016/j.jmathb.2019.100724>
- Singer, F. M., & Voica, C. (2008). Between perception and intuition: Learning about infinity. *The Journal of Mathematical Behavior*, *27*(3), 188–205. <https://doi.org/10.1016/j.jmathb.2008.06.001>
- Sutherland, M., Furjanic, D., Hermida, J., & Clarke, B. (2024). Using the number line to develop understanding of whole number magnitude and operations. *Intervention in School and Clinic*, *59*(3), 158–164. <https://doi.org/10.1177/10534512231156869>
- Tian, J., & Siegler, R. S. (2018). Which type of rational numbers should students learn first? *Educational Psychology Review*, *30*(2), 351–372. <https://doi.org/10.1007/s10648-017-9417-3>
- Vamvakoussi, X., & Vosniadou, S. (2004). Understanding the structure of the set of rational numbers: A conceptual change approach. *Learning and Instruction*, *14*(5), 453–467. <https://doi.org/10.1016/j.learninstruc.2004.06.013>
- Vamvakoussi, X., & Vosniadou, S. (2007). How many numbers are there in a rational numbers interval? Constraints, synthetic models and the effect of the number line. In S. Vosniadou, A. Baltas, & X. Vamvakoussi (Eds.), *Reframing the conceptual change approach in learning and instruction* (pp. 265–282). Elsevier.
- Vamvakoussi, X., & Vosniadou, S. (2010). How many decimals are there between two fractions? Aspects of secondary school students' understanding of rational numbers and their notation. *Cognition and Instruction*, *28*(2), 181–209. <https://doi.org/10.1080/07370001003676603>
- Vamvakoussi, X., & Vosniadou, S. (2012). Bridging the gap between the dense and the discrete: The number line and the “rubber line” bridging analogy. *Mathematical Thinking and Learning*, *14*(4), 265–284. <https://doi.org/10.1080/10986065.2012.717378>
- vanMarle, K., Chu, F. W., Mou, Y., Seok, J. H., Rouder, J. N., & Geary, D. C. (2018). Attaching meaning to the number words: Contributions of the object tracking and approximate number systems. *Developmental Science*, *21*(1), Article e12495. <https://doi.org/10.1111/desc.12495>
- Varma, S., Blair, K. P., & Schwartz, D. L. (2019). Cognitive science foundations of integer understanding and instruction. In A. Norton & M. W. Alibali (Eds.), *Constructing number: Merging perspectives from psychology and mathematics education* (pp. 307–327). Springer. https://doi.org/10.1007/978-3-030-00491-0_14
- Varma, S., & Schwartz, D. L. (2011). The mental representation of integers: An abstract-to-concrete shift in the understanding of mathematical concepts. *Cognition*, *121*(3), 363–385. <https://doi.org/10.1016/j.cognition.2011.08.005>
- VERBI Software. (2021). MAXQDA 2022 [Computer software]. Berlin, Germany: VERBI Software. Available from <https://www.maxqda.com>
- Vest, N., & Alibali, M. W. (2021). The mental representation of integers: Further evidence for the negative number line as a reflection of the natural number line. In *Proceedings of the Annual Meeting of the Cognitive Science Society* (Vol. 43, pp. 1670-1676). <https://escholarship.org/uc/item/8629819g>
- Viegut, A. A., & Matthews, P. G. (2023). Building fraction magnitude knowledge with number lines: Partitioning versus analogy. *Developmental Psychology*, *59*(10), Article 1757. <https://doi.org/10.1037/dev0001616>

- Vosniadou, S. (2021). Looking at the development of mathematical knowledge from the perspective of the framework theory approach to conceptual change: Lessons for mathematics education. In W. Fias & A. Henik (Eds.), *Heterogeneous contributions to numerical cognition* (pp. 95-115). Academic Press.
- Vosniadou, S., & Brewer, W. F. (1992). Mental models of the earth: A study of conceptual change in childhood. *Cognitive Psychology*, 24(4), 535–585. [https://doi.org/10.1016/0010-0285\(92\)90018-W](https://doi.org/10.1016/0010-0285(92)90018-W)
- Zorzi, M., Priftis, K., & Umiltà, C. (2002). Brain damage neglect disrupts the mental number line. *Nature*, 417(6885), 138–139. <https://doi.org/10.1038/417138a>

Appendix

Main question:

- This drawing represents a number line. Could you describe how it works in your own words?

Follow-up questions:

- Does zero have to be in the middle (if the student placed zero on the number line)?
- What do arrows represent (if the student did not mention infinity)?
- Can you elaborate on that (if the student's answer was not clear)?
- Could you talk a little bit more about ... (if the student's answer was not clear)?
- Is there anything you would like to add to the number line (if the student was hesitant about his/her number line explanation)?
- Can you show your explanations on the number line (if the student did not represent it in his/her initial drawing)?
- If you were to mark this number line, what would you do (if the number line was empty)?



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