PsychOpen GOLD

Empirical Research

Utilizing Analogical Reasoning to Aid Children's Proportional Reasoning Understanding

Lillian Ham[a], Elizabeth A. Gunderson*[a]

[a] Department of Psychology, Temple University, Philadelphia, PA, USA.

Abstract

Proportional reasoning is an important skill that relates to fraction learning and math achievement. Because both proportional and analogical reasoning involve comparing relations, we hypothesized that supports for analogical reasoning (multiple exemplars and labels) would help children match discrete proportions. Fourth and 5th graders (N = 119) completed a 16-item proportional equivalence choice task in a 2 (exemplars: one, two) x 3 (script type: juice mixing narrative, novel adjectives, no labels) x 2 (trial type: part-foil, whole-foil) mixed-effects design. The juice mixing script included common labels and a story paradigm, whereas the novel adjectives script only utilized common labels. The least-informative no-labels script served as a baseline. Results showed a significant three-way interaction between exemplars, script type, and trial type. Viewing two exemplars led to no significant differences across the other factors. However, when viewing one exemplar, children exhibited a part-matching bias with novel adjectives and no labels, but not the juice narrative. In sum, multiple exemplars and the juice narrative reduced children's part-matching bias; we conclude that supports for analogical reasoning can aid proportional reasoning as well.

Keywords: proportional reasoning, analogical reasoning, discrete quantities, multiple exemplars, common labeling, mathematical development

Using proportional reasoning to think relationally about quantities is highly relevant in everyday life. For example, if one pound of pasta serves four people, you may think proportionally to approximate how many pounds serves six people. Proportional reasoning is also valuable in developing and understanding math concepts. Previous studies have found proportional reasoning with continuous, part-whole representations to be correlated with fraction knowledge and later math achievement (Möhring, Newcombe, Levine, & Frick, 2016). Based on the Common Core State Standards for mathematics, children in the United States should be introduced to fractions around 3rd grade and continue developing fraction concepts (e.g., equivalence, adding, and subtracting) through 5th grade. By 6th and 7th grade, children should be able to analyze ratios and proportional relationships, using realistic examples to solve problems (e.g., constant speed, simple interest; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). As an important mathematical foundation that students often find challenging, proportional reasoning has been extensively researched and classroom interventions have been proposed (Berk, Taber, Gorowara, & Poetzl, 2009; Fujimura, 2001; Jitendra, Star, Rodriguez, Lindell, & Someki, 2011). Some neurocognitive building blocks such as the approxi-

mate number system have been identified to provide an early foundation for developing numerical concepts (Halberda & Feigenson, 2008; vanMarle et al., 2018). Likewise, a non-count based, ratio-processing system (RPS) which involves non-symbolic processing of ratios can serve as an intuitive basis for understanding proportions. In fact, performance on four RPS tasks using visual quantities such as line lengths or numbers of dots was found to predict later symbolic fraction understanding (Matthews, Lewis, & Hubbard, 2016). Similarly, when presented with continuous representations, infants and 4-year-olds are able to encode extent (height) in the presence of a standard, demonstrating this early, intuitive sense of proportions (Duffy, Huttenlocher, & Levine, 2005; Huttenlocher, Duffy, & Levine, 2002).

Despite an early-developing competency, school-age children struggle to focus on proportions when matching based on whole number quantities is an available strategy (Boyer, Levine, & Huttenlocher, 2008). For example, 6-year-olds can match equivalent proportions when they are presented in a continuous format; however, children at 10-years-old continue to struggle to match discrete proportions, which elicit an inclination to count-and-match rather than compare ratios (Boyer & Levine, 2015; Jeong, Levine, & Huttenlocher, 2007; Singer-Freeman & Goswami, 2001). Boyer and colleagues (2008) compared performance on a proportional equivalence choice task in which participants matched a target (continuous or discrete proportion) to one of two choices (both either continuous or discrete proportions). Children from kindergarten to 4th grade were least likely to choose the correct proportional match in the discrete-target and discrete-choices condition (Boyer et al., 2008). When directly comparing countable, discrete units, children implemented erroneous count-and-match strategies (e.g., matching 2/6 to 5/6 because both have equal total amounts, see Figure 1a; Boyer et al., 2008; Jeong et al., 2007).

There is also some evidence that erroneous count-and-match strategies are more prevalent when children can match the numerical quantities of parts of the proportion, versus the whole proportion – indicative of a part-matching bias. Boyer and colleagues (2008) presented elementary-aged children a target proportion (e.g., 2/6) and two answer choices; one choice was a proportional equivalent (e.g., 3/9), and the other choice matched based on either numerical parts (e.g., 2/3, a "part foil") or wholes (e.g., 4/6, a "whole foil"). Participants incorrectly matched based on parts more frequently than wholes (Boyer et al., 2008), corroborating the presence of a part-matching bias in young children. Children's focus on parts of proportions, and their focus on part-part relations over part-whole relations, has been noted in other studies (Singer & Resnick, 1992; Spinillo, 2002). Given these findings, we looked for potential differences in children's performance when part-matching was possible (part-foil trials, e.g., Figure 1b) and when it was not (whole-foil trials, e.g., Figure 1a).

Notably, the count-and-match error children often make when reasoning about discrete proportions is similar to the whole-number bias in formal fraction learning, i.e., the tendency to interpret a fraction's numerator and denominator as separate whole-number quantities instead of as a ratio (Ni & Zhou, 2005). In both cases, children compare proportions (or fractions) based on the numerical similarity of their components (i.e., their parts or the whole), rather than the relations between those components (Boyer et al., 2008; Braithwaite & Siegler, 2018); the primary difference is whether the parts and whole are represented as number symbols (in a fraction) or as visual countable units (in a discrete proportion). Given their conceptual similarity, research on how to reduce children's count-and-match bias in discrete proportional reasoning may have implications for reducing children's whole-number bias in fraction learning.

Here, we propose that methods proven to support analogical reasoning should also be effective in improving children's proportional reasoning. Analogical and proportional reasoning may evoke similar cognitive processes because both require reasoning about relational similarities, rather than perceptual similarities. For example, in a typical analogical reasoning task, children must determine whether a bicycle is more like a skateboard or eyeglasses; choosing the skateboard requires focusing on categorical commonalities rather than perceptual ones (Gentner & Namy, 1999). Determining whether a discrete visual representation of 3/4 is equivalent to 6/8 or 3/9 could require similar skills (Boyer et al., 2008; Möhring et al., 2016). Similar to the bicycle and eyeglasses, 3/4 is perceptually similar to 3/9 (i.e., both have a part quantity of 3 units), but they are proportionally distinct. Though somewhat perceptually similar, 3/4 and 6/8 are distinct in terms of their whole number quantities, but relationally equivalent.

Some prior work has shown links between children's analogical reasoning ability and their performance on proportional reasoning tasks (Boyer & Levine, 2015; Hurst & Cordes, 2018; Kotovsky & Gentner, 1996). In one study, 3- and 6-year-olds' performance on a pattern analogy task (e.g., yellow diamond:yellow circle::red diamond:?) was correlated with their performance on a continuous proportional reasoning task (a spinner comparison task in which children were asked to choose which of two spinners would increase their likelihood of winning stickers) (Hurst & Cordes, 2018). Further, two studies have found that completing trials using continuous proportions helped children to reason proportionally on subsequent trials using discrete proportions, although this was only the case for older children in each study (Boyer & Levine [2015]: 4th graders, but not kindergarten or 2nd graders on a proportional equivalence choice task; Hurst & Cordes [2018]: 5-6-year-olds, but not 3-4-year-olds on a spinner comparison task). The sequencing of trials from continuous to discrete is similar to another support for analogical reasoning, progressive alignment, in which identifying concrete relations makes it easier for children to identify subsequent abstract relations (Kotovsky & Gentner, 1996). Thus, these few existing studies provide tentative support for an empirical link between analogical and proportional reasoning.

Based on these theoretical and empirical considerations, we test the novel hypotheses that incorporating multiple exemplars and common labeling – evidence-based supports for analogical reasoning – will facilitate children's ability to reason proportionally about discrete units.

Prior work has identified two key supports for analogical reasoning: multiple exemplars and common labels. Studies have found that children who were shown multiple exemplars of a relational category were more successful in choosing the relational match than children who were shown only one exemplar (Gentner & Namy, 1999; Namy & Gentner, 2002). For example, Gentner and Namy (1999) asked 4-year-olds to choose another "blicket" after being shown a bicycle and/or a tricycle as exemplars. With only one exemplar (either the bicycle or tricycle), children selected the perceptually similar, but conceptually different match (eyeglasses). In contrast, when shown two exemplars (bicycle and tricycle), children selected the conceptually similar, but perceptually different match (skateboard). Further, two exemplars only helped children find relational commonalities when the exemplars shared relational similarities (Namy & Gentner, 2002; Waxman & Klibanoff, 2000). Based on this evidence, we hypothesize that aligning two proportionally equivalent, but perceptually different exemplars will help children find the relational commonality and select the proportional match.

In addition to multiple exemplars, common labeling has been shown to support analogical reasoning. Sharing labels between two objects encourages children to compare their commonalities, even when using novel labels such as "blicket" (Christie & Gentner, 2014; Gentner & Namy, 2006; Namy & Gentner, 2002; Waxman &

Klibanoff, 2000). For example, 4-year-olds who were shown two objects with the same label (i.e., "This is a blicket and this is also a blicket") chose the category or relational match (Namy & Gentner, 2002). In contrast, participants who were given conflicting labels (i.e., "This is a blicket and this is a daxen") chose the perceptual match (Namy & Gentner, 2002). These results indicate that young children can understand deeper relational commonalities when objects are given common labels. In this regard, we expect that providing common labels will prompt children to compare commonalities between discrete proportions, encouraging them to find a relation beyond numbers of units and helping them recognize relations of proportional equivalence.

Moreover, labeling a relation using a novel adjective (as opposed to a novel noun) may help offset a whole-object word-learning bias – the erroneous tendency to assume a novel label refers to an entire object as opposed to its properties or parts (Hollich et al., 2000). This word-learning bias can impact learning of mathematical concepts. For example, when children are shown two lines meeting at a single vertex (a "V" shape) and are told, "this is an angle", children assume that the word "angle" refers to the whole object (i.e., the V shape), rather than a particular property of the object (i.e., the degree of rotation), which can lead to misconceptions about the meaning of "larger angles" and "smaller angles" (Gibson, Congdon, & Levine, 2015). Similarly, using an adjective rather than a noun may reduce the ambiguity of the novel word's referent and help children associate the novel word with a property of the object (in this case, the proportional relation) rather than with the whole object. Adjectives can also work in conjunction with exemplars to facilitate comparison of properties across different objects (Waxman & Klibanoff, 2000). Waxman and Klibanoff (2000) found that when 3-year-olds were shown two different exemplars with a shared property (e.g., a transparent plate and a transparent toothbrush), both labeled by a novel adjective (i.e., "blick-ish"), they were able to correctly extend the novel adjective to other transparent objects. In contrast, children failed to do so if the exemplars were the same type of object or if the property was not labeled (Waxman & Klibanoff, 2000). Here, we expected novel adjective labels to facilitate children's ability to match proportions, when compared to no labels.

We were also interested in children's response to labeling proportions in a narrative context of mixing juice and water, a relatively complex referent often used in prior research (Boyer & Levine, 2015; Boyer et al., 2008; Möhring et al., 2016). One purpose for including the juice mixing script in the present study was to replicate and extend these findings. Another aim was to address the utility of the juice mixing narrative, that is, to determine if this descriptive script prompts children to reason proportionally more so than less descriptive scripts. The juice mixing script may help children avoid whole-object word-learning biases by providing different labels for the whole object ("glass of juice") and for both parts of the proportion ("juice" and "water"). Moreover, the conceptual relation between the terms "juice", "water", and "glass of juice" may present similar advantages to relational labels (e.g., "Daddy", "Mommy", "Baby") and invite children to compare relations rather than numerical quantities (Ratterman & Gentner, 1998). Encouraging children to think about a discrete representation continuously (i.e., mixing juice) may activate strategies based on continuous proportion estimation and help children avoid using incorrect count-and-match strategies. A conceptual referent, such as mixing juice and water, can be quite powerful in influencing how children think about proportions. For example, two different food analogies had disparate outcomes on a proportional matching task (Singer-Freeman & Goswami, 2001). Though both foods were depicted in a discrete manner, 3- and 4-year-olds had greater success reasoning proportionally about a pizza (a continuous conceptual referent) than a box of chocolates (or discrete referent) (Singer-Freeman & Goswami, 2001). For this reason, a continuous paradigm such as the juice mixing narrative may provide an advantageous context for reasoning about proportionality. Therefore, we expected the juice mixing narrative to

be more advantageous for proportional reasoning than no labels, which provide very little information of the task at hand.

Based on evidence that children struggle with discrete quantities (Boyer & Levine, 2015; Jeong et al., 2007; Singer-Freeman & Goswami, 2001) and because discrete representations are commonly associated with symbolic fractions (Rapp, Bassok, DeWolf, & Holyoak, 2015; Singer-Freeman & Goswami, 2001), the current study focused solely on proportional reasoning about discrete units. Further, we recruited 4th and 5th graders based on prior research showing that children this age continue to struggle on a discrete proportion matching task (Boyer & Levine, 2015). Drawing on a theoretical analysis of proportional reasoning and analogical reasoning, we asked: do multiple exemplars and common labeling encourage children to reason proportionally when comparing discrete quantities? Based on previous studies utilizing the juice and water mixing paradigm in proportional reasoning contexts, we also asked: does the juice mixing narrative help children to match equivalent proportions, over and above common labeling or no labels?

To address these questions, we compared proportional reasoning performance across two exemplar conditions (one-exemplar versus two-exemplars) and three script conditions (juice mixing, novel adjectives, and no labels), a 2 x 3 between-subjects design. We also manipulated foil type (part-foil, whole-foil) as a within-subjects factor. As noted previously, prior studies using the proportional equivalence choice task have shown that young children typically perform worse on part-foil than whole-foil items, indicating a part-matching bias (Boyer et al., 2008).

We hypothesized that children would make more proportional matches with two exemplars than one exemplar on the basis that aligning two exemplars encourages deep, relational thinking beyond perceptual commonalities. We also hypothesized that a juice mixing narrative (Boyer et al., 2008) and a novel adjectives condition, which both implement forms of common labeling, would yield more proportional matches than a no label condition. Our hypothesis regarding the relation between the juice narrative and the novel adjectives conditions was less clear. On the one hand, children could perform better in the juice narrative than the novel adjectives condition because the juice narrative is more elaborated, highlights continuity (i.e., mixing juice), and draws on prior knowledge by using commonplace, related terms (i.e., juice, water, mixing, taste). On the other hand, prior research shows that common labeling alone – even with novel adjectives – may be sufficient to direct attention to relational commonalities (Gentner & Namy, 1999, 2006; Namy & Gentner, 2002), potentially leading to no significant differences between the novel adjectives and juice mixing scripts.

It is also worth noting that, in the condition with no labels and a single exemplar, the task is ambiguous – a child could reasonably interpret the instruction, "Which one of these is just like this one?" as requesting a match based on proportion or based on numerical values. Thus, this condition serves as a baseline for children's tendency to match based on proportions versus numerical values of the part or whole. Thus, we can compare performance in the other conditions to this baseline to establish the impact of additional supports (novel labels, juice narrative, and/or multiple exemplars) on children's tendency to match based on proportions.

We expect the results of the present study to elucidate which, if any, analogical reasoning supports (multiple exemplars and labeling) are most beneficial in helping children overcome incorrect count-and-match strategies to reason about discrete proportions.

PsychOpen GOLD

Method

Participants

Children who received parental consent and assented to the study were enrolled. Participants were 119 children in 4th and 5th grades (M_{age} = 10.04 years, SD_{age} = 0.59; 64 girls; 78 4th graders) from eight schools in a large city in the Eastern United States. The average family income was $44,049 ($SD$ = $30,400, n = 71) and average parental education (maximum of both parents) was 14.0 years (SD = 2.47, n = 83). Of the participants whose parents reported their race/ethnicity (n = 86), 38.4% were Black or African American, 18.6% Hispanic, 17.4% White, 15.1% multiple race/ethnicities, 7.0% Asian/Asian American, 2.3% American Indian/Alaskan Native, and 1.2% Other.

Measure

Children completed a 16-item, computerized proportional equivalence choice task adapted from Boyer and colleagues (2008). The task displayed one or two exemplar proportions on the left side of the laptop screen and two choice proportions on the right side (see Figure 1).
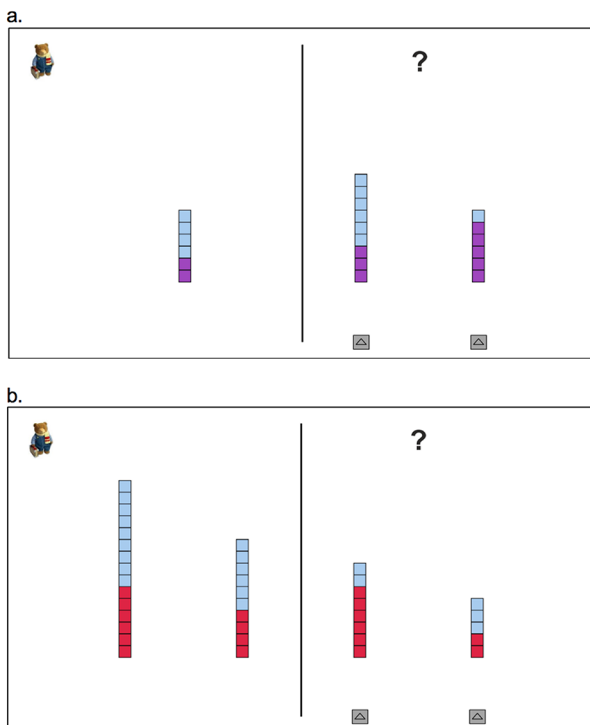

Figure 1. Example trial from one-exemplar condition (a). Example trial from two-exemplars condition (b). Adapted with permission from "Development of Proportional Reasoning: Where Young Children Go Wrong," by T. W. Boyer, S. C. Levine, and J. Huttenlocher, 2008, *Developmental Psychology*, *44*(5), p. 1478. Copyright 2008 by the American Psychological Association.

The task presented the exemplar(s) first and choices second, in order to encourage processing of the exemplar(s) before viewing the choices. All stimuli remained on the screen for the child to choose his/her answer. The exemplar and choice proportions were represented as columns of discrete units, demarcated by black

lines. Consistent with prior research using the juice mixing paradigm, the lower part units (colored units) were colored yellow, red, purple, green, or orange, and the upper part units were always light blue. A small picture of a teddy-bear named "Wally-bear" was displayed on the top left corner of the screen. Stimuli were identical across all script conditions. The one-exemplar and two-exemplars conditions differed only in whether one or two targets were displayed.

Each trial had two choices: the correct proportional match (e.g., 2 purple and 4 blue units matched 3 purple and 6 blue units, Figure 1a) and either a part-foil or whole-foil choice (see Table A1). On part-foil trials (8 trials, see Figure 1b for an example), the incorrect choice matched the target in terms of the number of colored units, whereas on whole-foil trials (8 trials, see Figure 1a for an example), the incorrect choice matched the target in terms of the total number of units (colored units plus light blue units). Targets A and B (in two-exemplars trials) and the correct proportional choice were counterbalanced for side of presentation (i.e., left or right). Trials were presented in a random order. Scores were calculated as percent correct.

Design and Procedure

Participants were randomly assigned, within classroom, to one of six between-subjects conditions in a 2 (number of exemplars) x 3 (script type) design. Foil type was manipulated within subjects. In the one-exemplar conditions (Figure 1a), participants were shown one exemplar and two choices; in the two-exemplars conditions (Figure 1b), they were shown two exemplars of the same proportion and two choices (see Appendix 1 for all trials).

The three scripts (juice mixing, novel adjectives, and no label) differed in how the task was introduced and how the proportions were verbally labeled on each trial (see Appendix 2 for scripts). In all conditions, the experimenter introduced participants to "Wally-bear" (Boyer et al., 2008). In the juice mixing conditions, Wally-bear was described as a character who mixed various amounts of juice with water. On each trial, participants were asked (two-exemplars script in parentheses): "Wally says this glass of juice tastes just right. (He says this one also tastes just right). Which glass of juice has just the right amount of juice and just the right amount of water so that it tastes just like this one (these)?" In the novel adjectives condition, participants were asked (two-exemplars script in parentheses): "Wally says this is a very *dak-ish* one. (He says this one is also very *dak-ish*). Now Wally wants another one that is *dak-ish*. Can you give him another one that is *dak-ish*?" A different novel adjective was used for every trial. In the no label condition, participants were asked (two-exemplars script in parentheses): "Wally likes this one. (He also likes this one). Now Wally wants another one. Which one of these is just like this one (these)?"

Children were tested individually at their school by a trained experimenter. The proportional equivalence choice task was administered on a laptop while the experimenter read the script corresponding to the assigned condition. No feedback was given.
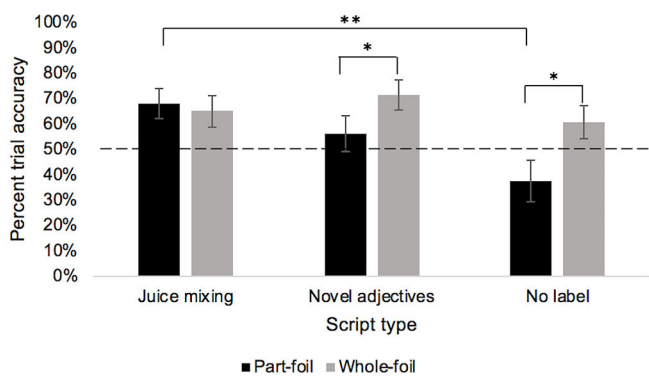
Results

Preliminary Analyses

Across conditions, performance was above chance, $M = .63$, $SD = .25$, $t(118) = 5.60$, $p < .001$. Fifth graders ($M = .70$, $SD = .23$) performed significantly better than 4th graders ($M = .59$, $SD = .25$), $t(117) = 2.38$, $p = .019$, $d = .46$. Both grade levels performed above chance, $ps < .01$. Performance did not differ as a function of gender (girls: $M = .62$, $SD = .26$; boys: $M = .64$, $SD = .25$, $t(117) = .45$, $p = .656$, $d = .08$). In addition, a preliminary ANOVA including gender and grade as factors found no significant interactions of these factors with our manipulated variables, $0.01 < F < 2.80$, $ps > .05$, $.001 < \eta_p^2 < .029$; therefore, we did not include grade or gender in our main analyses.

Main Analyses

We conducted a mixed-effects ANOVA on accuracy, with exemplar (one, two) and script type (juice mixing, novel adjectives, no labels) as between-subjects factors and foil type (part-foil, whole-foil) as a within-subjects factor (Figure 2; Table 1).
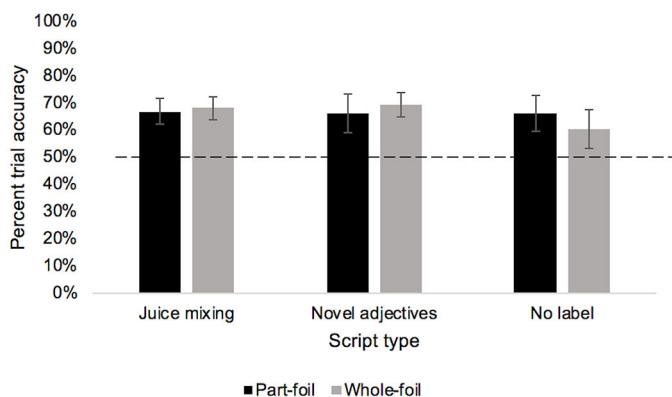
a.

b.

Figure 2. Percent accuracy by exemplar, script type, and foil type. One-exemplar conditions (a). Two-exemplars conditions (b). Error bars indicate one standard error of the mean. Dashed lines indicate chance performance (50%).

*$p < .05$. **$p < .01$.

There were no significant main effects of exemplar, $F(1, 113) = 1.89$, $p = .172$, $\eta_p^2 = .02$ or script type, $F(2, 113) = 2.35$, $p = .100$, $\eta_p^2 = .04$. However, there was a significant three-way interaction between exemplar, script type, and foil type, $F(2, 113) = 4.28$, $p = .016$, $\eta_p^2 = .07$. There was also a significant main effect of foil type, $F(1, 113) = 5.67$, $p = .019$, $\eta_p^2 = .05$, and a significant interaction between foil type and exemplar, $F(1, 113) = 6.78$, $p = .010$, $\eta_p^2 = .06$. The other two-way interactions were not significant (exemplar x script, $F(2, 113) = .77$, $p = .467$, $\eta_p^2 = .01$; foil type x script, $F(2, 113) = 1.94$, $p = .148$, $\eta_p^2 = .03$).

Effects Within One-Exemplar Conditions

To further understand this 3-way interaction, we first examined performance within the one-exemplar conditions (Figure 2a). Within the one-exemplar conditions, we conducted a 3 (script type) x 2 (foil type) mixed-effects AN-OVA. There was a significant main effect of foil type, $F(1, 55) = 9.73$, $p = .003$, $\eta_p^2 = .15$, but not script type, $F(2, 55) = 2.67$, $p = .078$, $\eta_p^2 = .09$, and a significant interaction between foil type and script type, $F(2, 55) = 4.40$, $p = .017$, $\eta_p^2 = .14$.

Table 1

Demographic Characteristics and Percent Accuracy by Condition (Exemplar, Script Type) and Foil Type

	Juice narrative		Novel adjectives		No label	
	One exemplar	Two exemplars	One exemplar	Two exemplars	One exemplar	Two exemplars
Measure	(*n* = 20)	(*n* = 20)	(*n* = 18)	(*n* = 20)	(*n* = 20)	(*n* = 21)
Demographics						
n_{girls}, n_{boys}	9, 11	12, 8	10, 8	11, 9	10, 10	12, 9
n_{4th}, n_{5th} Grade	14, 6	14, 6	11, 7	12, 8	13, 7	14, 7
M_{age} in years (*SD*)	9.99 (.72)	9.94 (.43)	10.08 (.61)	10.31 (.61)	9.92 (.54)	10.00 (.55)
Proportional reasoning task performance, % accuracy (*SD*)						
Total accuracy	.67 (.26)**	.68 (.20)***	.64 (.24)*	.68 (.25)**	.49 (.27)	.63 (.28)*
Part-foil trials accuracy	.68 (.27)**	.67 (.22)**	.56 (.30)	.66 (.32)*	.38 (.36)	.66 (.30)*
Whole-foil trials accuracy	.65 (.28)*	.68 (.19)***	.72 (.25)**	.69 (.20)***	.61 (.29)	.60 (.33)

*p < .05. **p < .01. ***p < .001. Asterisks indicate p-values from a one-sample t-test comparing accuracy to chance (50%).

Next, we examined the effect of foil type within the one-exemplar conditions, separately for each script type. In the one-exemplar, no label condition (a baseline with ambiguous instructions), children performed significantly worse on part-foil than whole-foil trials, $t(19) = 2.85$, $p = .010$, $d = .64$. Children also had worse performance on part-foil than whole-foil trials in the one-exemplar, novel adjectives condition, $t(17) = 2.35$, $p = .031$, $d = .55$. In contrast, there was no significant effect of foil type in the one-exemplar, juice mixing condition, $t(19) = .72$, $p = .480$, $d = .16$.

We also directly compared performance in the baseline one-exemplar, no label condition to the other one-exemplar conditions, to assess whether the juice narrative or novel adjectives led to greater proportional matching. Children's overall accuracy was significantly higher in the one-exemplar juice-mixing narrative condition than in the baseline one-exemplar, no labels condition, $t(38) = 2.10$, $p = .042$, $d = .68$; this difference was driven by significantly better performance on part-foil trials, $t(38) = 3.04$, $p = .004$, $d = .99$. In addition, overall accuracy was somewhat higher in the one-exemplar, novel adjectives condition, than in the baseline one-exemplar, no labels condition, though this difference was not statistically significant, $t(36) = 1.78$, $p = .083$, $d = .59$.

Effects Within Two-Exemplar Conditions

In contrast to the one-exemplar conditions, within the two-exemplars conditions (Figure 2b), there were no significant main effects of foil type, $F(1, 58) = .03$, $p = .856$, $\eta_p^2 = .001$, script type, $F(2, 58) = .25$, $p = .784$, $\eta_p^2 = .01$, or interactions of script type and foil type, $F(2, 58) = .93$, $p = .400$, $\eta_p^2 = .03$. Thus, script type and foil type impacted performance when one exemplar was shown, but did not significantly impact performance when two exemplars were provided.

Effects Within Part-Foil Trials

In order to understand the effects of the between-subjects manipulations on the more difficult part-foil trials, we also conducted follow-up ANOVAs within foil type. On part-foil trial trials, a 2 (number of exemplars) x 3 (script type) between-subjects ANOVA revealed a significant main effect of exemplar, $F(1, 113) = 5.16$, $p = .025$, $\eta_p^2 = .04$, no significant main effect of script type, $F(2, 113) = 2.85$, $p = .062$, $\eta_p^2 = .05$, and no significant interaction between script type and exemplar, $F(2, 113) = 2.58$, $p = .080$, $\eta_p^2 = .04$. The main effect of exemplar indicated that on part-foil trials, performance was significantly better with two exemplars ($M = .66$, $SD = .28$) than with one exemplar ($M = .54$, $SD = .33$), $t(117) = 2.22$, $p = .028$, $d = .41$.

Effects Within Whole-Foil Trials

On whole-foil trials, a 2 (number of exemplars) x 3 (script type) between-subjects ANOVA found no significant main effects of script type, $F(2, 113) = 1.50$, $p = .228$, $\eta_p^2 = .026$, exemplar, $F(1, 113) = .001$, $p = .974$, $\eta_p^2 < .001$, or interactions of script type and exemplar, $F(2, 113) = .10$, $p = .901$, $\eta_p^2 = .002$.

In sum, the presence of two exemplars led to higher performance specifically on part-foil trials, leading to no foil-type effect within any of the two-exemplars conditions. Notably, foil type did not significantly impact performance in any of the juice mixing conditions (even with only one exemplar). However, children performed worse on part-foil than whole-foil trials in the two conditions in which there was neither a juice mixing narrative nor multiple exemplars (i.e., one-exemplar with novel adjectives, and one-exemplar with no labels).

Discussion

Our results show that multiple exemplars and common labeling – supports that have been shown to aid analogical reasoning – can also support proportional reasoning in late elementary school. Although we had predicted overall benefits (i.e., main effects) of multiple exemplars and scripts that included common labeling (juice narrative and novel labels), our results were more nuanced and depended on trial type (a three-way interaction between exemplar, script type, and foil type with a small-to-medium effect, $\eta_p^2 = .07$). Specifically, a strong part-matching bias (i.e., a medium-to-large difference in accuracy between part-foil and whole-foil trials, $d = .64$) appeared in ambiguous contexts (i.e., one exemplar with no labels) when children were simply asked which images were alike and could match based on numerical quantity or proportional equivalence. However, this part-matching bias was eliminated when two exemplars were provided. This may be because showing children two examples of a proportion, which always differed in the numerical quantities of the part and whole (e.g., 4/16 and 3/12), helped children focus on relations rather than countable quantities, and therefore helped children avoid choosing based on the perceptual similarity between the exemplar and the part-foil choice (i.e., the number of countable, colored units).

In addition, the juice mixing narrative, which incorporates common labeling ("juice", "water") and also draws on conceptual knowledge about mixing continuous quantities, was effective in promoting proportional reasoning over and above baseline performance with a medium-to-large effect on total accuracy (one-exemplar juice narrative versus no-label conditions, d = .68). The juice mixing narrative also buffered against the part-matching bias that was elicited in ambiguous contexts. This provides support for the theory that the narrative script impacts proportional reasoning. Surprisingly, providing two exemplars and the juice mixing narrative together did not have an additive effect on performance. In other words, showing either two exemplars or applying the juice paradigm helped children think proportionally; however, providing both did not lead to additional benefits.

We also tested whether common labeling alone would improve proportional reasoning via the novel adjectives conditions (e.g., "this one is dak-ish"). With only one exemplar, novel adjectives led to performance that was marginally better than no labels, a medium-to-large effect (d = .59). In addition, the part-matching bias was still present in the one-exemplar, novel adjective condition with a medium-to-large effect (d = .55). These results indicate that novel adjectives alone were not sufficient to eliminate the part-matching bias. It is possible that the one-exemplar novel adjectives condition was ambiguous in the same way that the baseline, no labels condition was – although we expected children to interpret the novel adjective (e.g., "dak-ish") as describing the property of proportionality, some children may have interpreted it as relating to the numerical quantity of the parts or wholes. Nevertheless, the marginal improvement compared to the no labels condition, with a medium-to-large effect size (d = .59), was consistent with our prediction that novel adjectives would encourage proportional thinking. Future research replicating these findings could lead to greater clarity regarding the impact of novel adjectives alone on proportional reasoning.

The part-matching bias we observed in this study is consistent with previous findings that young children struggled more on part-foil trials than whole-foil trials (Boyer et al., 2008). As discussed by Boyer and colleagues (2008), this could be due to the perceptual salience of the numerator, as it was the only color that changed from trial to trial. Additionally, children may have a more general bias to reason part-to-part rather than part-to-whole when thinking relationally (Singer & Resnick, 1992). However, whereas prior work indicated that this part-matching bias disappeared by 4th grade (Boyer et al., 2008), the current study observed poorer performance on part-foil items than whole-foil items in 4th and 5th graders, but only in those conditions that did not provide multiple exemplars or a juice mixing narrative. Without either support, this part-matching bias, previously observed in younger children with the juice narrative, emerges at this older age. This finding raises the possibility that the juice mixing narrative may be as uninformative to 1st and 2nd graders as the no label condition is to 4th and 5th graders, if younger children have less familiarity than older children with the juice mixing context, or have more difficulty applying the notion of continuity (i.e., mixing juice) to reason proportionally. In fact, the juice mixing script is similar in complexity to a word problem, requiring the participant to first understand the grammatical rules and semantics of English in order to highlight key information and implement a strategy to find the solution (Jitendra et al., 2011), potentially making it difficult for young children to benefit from the juice mixing narrative. Future research could test younger children to determine whether 1st and 2nd graders benefit from the juice mixing narrative and presentation of multiple exemplars in the same way that 4th and 5th graders did in the present study. Prior research suggests that younger children may not benefit from such supports; for example, 4th graders benefitted from viewing continuous proportion matching trials before discrete ones, but kindergarteners and 2nd graders did not (Boyer & Levine, 2015). It may not be until 3rd or 4th grade that children can understand how to use such supports for analogical reasoning to overcome maladaptive count-and-match strategies.

These results are consistent with prior research showing that generally, children perform poorly on the proportional equivalence choice task when presented with a discrete target and discrete choices. On the same version of the task (i.e., one exemplar, juice mixing narrative), 4th graders in Boyer and colleagues (2008) performed with 66% accuracy on both part- and whole-foil trials, while our 4th and 5th graders performed with 68% accuracy on part-foil trials and 65% accuracy on whole-foil trials. While supports from analogical reasoning may improve understanding of matching discrete proportions, no manipulation in this study resulted in performance that was as high as performance involving continuous proportions in prior studies. For example, 4th graders shown a continuous target and discrete choices (Boyer et al., 2008) or eight continuous trials before eight discrete trials (Boyer & Levine, 2015) obtained accuracy levels of 80% or higher.

To our knowledge, this is the first study to show that multiple exemplars and script context (i.e., the juice narrative) improve children's ability to match discrete proportions. Moreover, our participants are older than children in previous analogical reasoning studies, extending evidence for the benefits of multiple exemplars and common labeling to a novel domain (proportional reasoning) and age group. Taken together, the implications of this research can inform classroom instruction and improve the teaching and learning of proportions and fractions, showing that young children with strong whole-number biases may benefit from analogical reasoning supports to overcome those biases and reason relationally.

Funding

Competing Interests

The authors have declared that no competing interests exist.

Acknowledgments

Data Availability

For this study, the data is freely available (see the Supplementary Materials section).

Supplementary Materials

The data for this project are available on Open Science Framework at https://osf.io/6jbme

Index of Supplementary Materials

Ham, L., & Gunderson, E. A. (2018). *Utilizing analogical reasoning to aid children's proportional reasoning understanding* [Data]. OSF. https://osf.io/6jbme

References

Berk, D., Taber, S. B., Gorowara, C. C., & Poetzl, C. (2009). Developing prospective elementary teachers' flexibility in the domain of proportional reasoning. *Mathematical Thinking and Learning, 11*(3), 113-135. https://doi.org/10.1080/10986060903022714

Boyer, T. W., & Levine, S. C. (2015). Prompting children to reason proportionally: Processing discrete units as continuous amounts. *Developmental Psychology, 51*(5), 615-620. https://doi.org/10.1037/a0039010

Boyer, T. W., Levine, S. C., & Huttenlocher, J. (2008). Development of proportional reasoning: Where young children go wrong. *Developmental Psychology, 44*(5), 1478-1490. https://doi.org/10.1037/a0013110

Braithwaite, D. W., & Siegler, R. S. (2018). Developmental changes in the whole number bias. *Developmental Science, 21*(2), Article e12541. https://doi.org/10.1111/desc.12541

Christie, S., & Gentner, D. (2014). Language helps children succeed on a classic analogy task. *Cognitive Science, 38*(2), 383-397. https://doi.org/10.1111/cogs.12099

Duffy, S., Huttenlocher, J., & Levine, S. (2005). It is all relative: How young children encode extent. *Journal of Cognition and Development, 6*(1), 51-63. https://doi.org/10.1207/s15327647jcd0601_4

Fujimura, N. (2001). Facilitating children's proportional reasoning: A model of reasoning processes and effects of intervention on strategy change. *Journal of Educational Psychology, 93*(3), 589-603. https://doi.org/10.1037/0022-0663.93.3.589

Gentner, D., & Namy, L. L. (1999). Comparison in the development of categories. *Cognitive Development, 14*(4), 487-513. https://doi.org/10.1016/S0885-2014(99)00016-7

Gentner, D., & Namy, L. L. (2006). Analogical processes in language learning. *Current Directions in Psychological Science, 15*(6), 297-301. https://doi.org/10.1111/j.1467-8721.2006.00456.x

Gibson, D. J., Congdon, E. L., & Levine, S. C. (2015). The effects of word-learning biases on children's concept of angle. *Child Development, 86*(1), 319-326. https://doi.org/10.1111/cdev.12286

Halberda, J., & Feigenson, L. (2008). Developmental change in the acuity of the "number sense": The approximate number system in 3-, 4-, 5-, and 6-year-olds and adults. *Developmental Psychology, 44*(5), 1457-1465. https://doi.org/10.1037/a0012682

Hollich, G. J., Hirsh-Pasek, K., Golinkoff, R. M., Brand, R. J., Brown, E., Chung, H. L., . . . Bloom, L. (2000). Breaking the language barrier: An emergentist coalition model for the origins of word learning. *Monographs of the Society for Research in Child Development, 65*(3), i-vi, 1-135.

Hurst, M. A., & Cordes, S. (2018). Attending to relations: Proportional reasoning in 3- to 6-year-old children. *Developmental Psychology, 54*(3), 428-439. https://doi.org/10.1037/dev0000440

Huttenlocher, J., Duffy, S., & Levine, S. (2002). Infants and toddlers discriminate amount: Are they measuring? *Psychological Science, 13*(3), 244-249. https://doi.org/10.1111/1467-9280.00445

PsychOpen GOLD

Jeong, Y., Levine, S. C., & Huttenlocher, J. (2007). The development of proportional reasoning: Effect of continuous versus discrete quantities. *Journal of Cognition and Development, 8*(2), 237-256. https://doi.org/10.1080/15248370701202471

Jitendra, A. K., Star, J. R., Rodriguez, M., Lindell, M., & Someki, F. (2011). Improving students' proportional thinking using schema-based instruction. *Learning and Instruction, 21*(6), 731-745. https://doi.org/10.1016/j.learninstruc.2011.04.002

Kotovsky, L., & Gentner, D. (1996). Comparison and categorization in the development of relational similarity. *Child Development, 67*(6), 2797-2822. https://doi.org/10.2307/1131753

Matthews, P. G., Lewis, M. R., & Hubbard, E. M. (2016). Individual differences in nonsymbolic ratio processing predict symbolic math performance. *Psychological Science, 27*(2), 191-202. https://doi.org/10.1177/0956797615617799

Möhring, W., Newcombe, N. S., Levine, S. C., & Frick, A. (2016). Spatial proportional reasoning is associated with formal knowledge about fractions. *Journal of Cognition and Development, 17*(1), 67-84. https://doi.org/10.1080/15248372.2014.996289

Namy, L. L., & Gentner, D. (2002). Making a silk purse out of two sow's ears: Young children's use of comparison in category learning. *Journal of Experimental Psychology: General, 131*(1), 5-15. https://doi.org/10.1037/0096-3445.131.1.5

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common Core State Standards for Mathematics*. Retrieved from http://www.corestandards.org/wp-content/uploads/Math_Standards1.pdf

Ni, Y., & Zhou, Y.-D. (2005). Teaching and learning fraction and rational numbers: The origins and implications of whole number bias. *Educational Psychologist, 40*(1), 27-52. https://doi.org/10.1207/s15326985ep4001_3

Rapp, M., Bassok, M., DeWolf, M., & Holyoak, K. J. (2015). Modeling discrete and continuous entities with fractions and decimals. *Journal of Experimental Psychology: Applied, 21*(1), 47-56. https://doi.org/10.1037/xap0000036

Ratterman, M. J., & Gentner, D. (1998). The effect of language on similarity: The use of relational labels improves young children's performance in a mapping task. In K. Holyoak, D. Gentner, & B. Kokinov (Eds.), *Advances in analogy research: Integration of theory and data from the cognitive, computational, and neural sciences* (pp. 274–282). Sofia, Bulgaria: New Bulgarian University.

Singer, J. A., & Resnick, L. B. (1992). Representations of proportional relationships: Are children part-part or part-whole reasoners? *Educational Studies in Mathematics, 23*(3), 231-246. https://doi.org/10.1007/BF02309531

Singer-Freeman, K. E., & Goswami, U. (2001). Does half a pizza equal half a box of chocolates? Proportional matching in an analogy task. *Cognitive Development, 16*(3), 811-829. https://doi.org/10.1016/S0885-2014(01)00066-1

Spinillo, A. G. (2002). Children's use of part–part comparisons to estimate probability. *The Journal of Mathematical Behavior, 21*(3), 357-369. https://doi.org/10.1016/S0732-3123(02)00134-7

vanMarle, K., Chu, F. W., Mou, Y., Seok, J. H., Rouder, J., & Geary, D. C. (2018). Attaching meaning to the number words: Contributions of the object tracking and approximate number systems. *Developmental Science, 21*(1), Article e12495. https://doi.org/10.1111/desc.12495

Waxman, S. R., & Klibanoff, R. S. (2000). The role of comparison in the extension of novel adjectives. *Developmental Psychology, 36*(5), 571-581. https://doi.org/10.1037/0012-1649.36.5.571

Appendices

Appendix 1: Trials

Fraction numerators represent the colored units of the proportion. Denominators represent the total number of units of the proportion. The Foil matched Target A based on numerator (part-foil) or denominator (whole-foil). In the one-exemplar conditions, only Target A, Match, and Foil were displayed. In the two-exemplars conditions, Target A, Target B, Match, and Foil were displayed. Targets (A and B) and choices (match and foil) were counterbalanced for side of presentation (i.e., left or right). Trial order was randomized across participants. Disparity = | proportional match – foil |.

Table A1

Proportional Equivalence Choice Task Trials by Foil Type

Trial	Target A	Target B	Match	Foil	Disparity
Part-foils					
1	4/5	8/10	12/15	4/12	.47
2	4/12	1/3	3/9	4/5	.47
3	4/16	3/12	2/8	4/6	.42
4	6/8	3/4	9/12	6/15	.35
5	6/15	4/10	2/5	6/8	.35
6	2/6	4/12	3/9	2/3	.33
7	4/6	2/3	6/9	4/12	.33
8	4/6	6/9	2/3	4/12	.33
Whole-foils					
9	2/6	4/12	3/9	5/6	.50
10	10/15	2/3	4/6	3/15	.47
11	8/12	2/3	4/6	3/12	.42
12	2/10	3/15	1/5	6/10	.40
13	6/10	3/5	9/15	2/10	.40
14	2/6	1/3	5/15	4/6	.33
15	2/6	1/3	3/9	4/6	.33
16	6/9	8/12	2/3	3/9	.33

Appendix 2: Scripts

Juice Mixing Script

Point to the photo of Wally-bear and begin by saying:

> *"His name is Wally-bear, and Wally-bear really, really likes drinking juice. Wally-bear loves orange juice, but he also loves all other kinds of juice – he loves red juice, blue juice, green juice, yellow juice, and purple juice – all kinds of juice. Wally-bear loves juice so much that sometimes he likes to mix his juice himself, but he needs to be really careful when he's mixing his juice, so he has just enough juice and just enough water, so that it tastes just right! Let me show you."*

Click the button so that an orange and blue column depicting two juice units and two water units appears on the left.

"See, when he was mixing this glass of orange juice [pointing to the column]*, Wally needed just the right amount of orange* [gesturing to the orange portion of the column]*, and he needed just the right amount of water* [gesturing to the light blue portion of the column]*, so that when he mixed them up* [gesturing with a circular motion around the entire column]*, it would taste just right. Let me show you what would happen if Wally wanted more juice."*

Click the button so that an orange and blue column depicting three juice units and three water units appears on the right.

"See, if Wally wanted more juice, he would need to mix more orange [pointing to the orange portion of the secondary column, and gesturing back and forth between the initial and secondary orange amounts]*, and he would need more water* [pointing to the light blue portion of the secondary column, and gesturing back and forth between the initial and secondary water amounts]*, so that when he mixes his juice* [gesturing with a circular motion around the entire secondary column]*, it would still taste just right* [gesturing with a circular motion around the entire initial column]*, just like it is supposed to* [with a point to the glass of juice the character appears to be holding]*."*

Click the button so that both the columns disappear.

"Now I have a question for you. You see, Wally knows how to mix his juice, and he knows what the different kinds of juice are supposed to taste like, but sometimes he gets a little confused, and he doesn't know how much juice and how much water he needs to mix for his different kinds of juice. Do you think you could help him by picking out the right mixes for his different kinds of juice? Do you think you can pick out the right mixes so that Wally's juice tastes just right?"

For one exemplar:

Click the button so that the exemplar appears. Prompt the participant:

"Wally says this glass of juice tastes just right."

Press the spacebar so that the two choices appear. Prompt the participant:

"Which glass of juice [pointing to the choices] *has just the right amount of juice and just the right amount of water so that it tastes just like this one* [pointing to the exemplar]*?* Click on the button below the glass of juice that tastes just right.*"*

Do not give feedback.

For two exemplars:

Click the button so that the two exemplars appear. Prompt the participant:

"Wally says this glass of juice tastes just right [pointing to the exemplar on the left]*. He says this one also tastes just right* [pointing to the exemplar on the right]*."*

Press the spacebar so that the two choices appear. Prompt the participant:

"Which glass of juice [pointing to the choices] *has just the right amount of juice and just the right amount of water so that it tastes just like these* [pointing to the exemplars]*?* Click on the button below the glass of juice that tastes just right.*"*

Do not give feedback.

Novel Adjectives Script

Point to the photo of Wally-bear and begin by saying:

"His name is Wally-bear."

Click the button so that an orange and blue column depicting two orange units and two blue units appears on the left.

"Wally says this is a very blick-ish one [pointing to the column]*."*

Click the button so that an orange and blue column depicting three orange units and three blue units appears on the right.

"He says this one is also very blick-ish [pointing to the column on the right]*."*

Click the button so that both the columns disappear.

"Now I have a question for you. Do you think you could help Wally by picking out the right ones?"

For one exemplar:

Click the button so that the exemplar appears. Prompt the participant:

"Wally says this is a very [novel adjective] *one."*

Press the spacebar so that the two choices appear. Prompt the participant:

"Can you give him another one that is [novel adjective]*? <u>Click on the button below the one that is</u>* [novel adjective]*.*"

Do not give feedback.

For two exemplars:

Click the button so that the two exemplars appear. Prompt the participant:

"Wally says this is a very [novel adjective] *one* [pointing to the exemplar on the left]*. He says this one is also very* [novel adjective] [pointing to the exemplar on the right]*."*

Press the spacebar so that the two choices appear. Prompt the participant:

"Now Wally wants another one that is [novel adjective]*. Can you give him another one that is* [novel adjective]*? <u>Click on the button below the one that is</u>* [novel adjective]*."*

Do not give feedback.

Novel adjectives (16): dak-ish, zav-ish, wugg-ish, fepp-ish, tal-ish, bem-ish, hux-ish, tand-ish, gloop-ish, dand-ish, foom-ish, bink-ish, vab-ish, dupe-ish, wolp-ish, kern-ish

No Label Script

Point to the photo of Wally-bear and begin by saying:

"His name is Wally-bear."

Click the button so that an orange and blue column depicting two orange units and two blue units appears on the left.

"Wally likes this one [pointing to the column]*."*

Click the button so that an orange and blue column depicting three orange units and three blue units appears on the right.

"Here is another one [pointing to the column on the right] *that is just like this one* [pointing to the column on the left]*."*

Click the button so that both the columns disappear.

"Now I have a question for you. Do you think you could help Wally by picking out the ones he likes?"

For one exemplar:

Click the button so that the exemplar appears. Prompt the participant:

"Wally likes this one."

Press the spacebar so that the two choices appear. Prompt the participant:

"Now Wally wants another one. Which one of these [pointing to the choices] *is just like this one* [pointing to the exemplar]*? <u>Click on the button below the one that is just like this one</u>* <u>[pointing to the exemplar]</u>*."*

Do not give feedback.

<u>For two exemplars:</u>

Click the button so that the two exemplars appear. Prompt the participant:

"Wally likes this one [pointing to the exemplar on the left]*. He also likes this one* [pointing to the exemplar on the right]*."*

Press the spacebar so that the two choices appear. Prompt the participant:

"Now Wally wants another one. Which one of these [pointing to the choices] *is just like these* [pointing to the exemplars]*? <u>Click on the button below the one that is just like these</u>* <u>[pointing to the exemplars]</u>*."*

Do not give feedback.